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A New Solution for the 1D Radiative Transfer Equation

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A new solution to the angularly discretized radiative transfer equation in a 1D slab medium with anisotropic scattering has been proposed. While similar to other solutions involving linear algebra, the proposed solution avoids instability caused by stiffness by expressing the complimentary part in terms of the hyperbolic sinh rather than simple exponentials. The effectiveness of this solution to produce extreme benchmark quality reflectances and transmittances with Wynn-epsilon convergence acceleration is demonstrated.

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INTRODUCTION

The 2N stream, directionally (μ) discretized equation of radiative transfer,

$$\left[\mu_m \frac{\partial}{\partial \tau} + 1\right] I_m(\tau) = \omega \sum_{m'=1}^{2N} \alpha_{m'} p(\mu_m, \mu_{m'}) I_{m'}(\tau),\ m = 1, 2, ..., 2N, \qquad (1)$$

has been the subject of various numerical algorithms over the past 65 years since Chandrasekhar [1] first introduced it. The importance of this equation cannot be overemphasized, as it characterizes neutral particles (in this case photons) undergoing collisions from direction $\mu_{m'}$ to direction μ_m, as characterized by the scattering phase function, $p(\mu_m, \mu_{m'})$, in a one-dimensional (τ) slab medium. More precisely, particles enter a homogeneous slab medium of width τ_0 to establish the particle intensity distribution $I_m(\tau)$. We find applications in light scattering for graphical rendering as well as neutron transport in a scattering/fissioning medium. Noting the differential/algebraic nature of the equation, numerous attempts at an efficient numerical algorithm have appeared in the literature-- many naturally involving numerical linear algebra. Most notably among the methods developed are the Discrete Ordinates Method [DOM] of Stamnes [2], the Analytical Discrete Ordinates (ADO) method of Siewert [3] as well as the Laplace Transform SN (LTSN) method of Vilhena [4]. The methods are all centered on the determination of eigenvalues (or zeros of a polynomial for the LTSN method) of a linear system enabling the solution to be represented analytically as an exponential sum. Because of the stiff nature of the Jacobian however, unless care is taken, such a representation leads to instability. Here, we derive an alternative form of solution to overcome this difficulty.

The basis of the linear algebra approach is the reformulation of Eq(1) in terms of positively and negatively directed photons to give

$$\frac{d\boldsymbol{I}^{\pm}(\tau)}{d\tau} = \mp \boldsymbol{M}^{-1}\left(\boldsymbol{I}_N - \boldsymbol{C}^{\pm\pm}\right)\boldsymbol{I}^{\pm}(\tau) + \boldsymbol{M}^{-1}\boldsymbol{C}^{\pm\mp}\boldsymbol{I}^{\mp}(\tau),$$

where

$$\begin{aligned} &\boldsymbol{M} \equiv diag\left\{\mu_m\right\} \\ &\boldsymbol{I}^{\mp}(\tau) \equiv \begin{bmatrix} I_{\left\{ 1 \atop N+1 \right\}} & I_{\left\{ 2 \atop N+2 \right\}} & \ldots & I_{\left\{ N \atop 2N \right\}} \end{bmatrix}^T \\ &\boldsymbol{C} \equiv \begin{bmatrix} \boldsymbol{C}^{--} & \boldsymbol{C}^{-+} \\ \boldsymbol{C}^{+-} & \boldsymbol{C}^{++} \end{bmatrix} = \left\{ \frac{\omega}{2}\alpha_m \sum_{l=0}^{L} \omega_l P_l\left(\pm\mu_j\right) P_l\left(\pm\mu_m\right);\; j,m = 1,\ldots N \right\}, \end{aligned} \tag{2}$$

and $\boldsymbol{I}_N$ is the identity matrix of order N with L the order of the truncated Legendre polynomial expansion of the scattering phase function. Then, by forming the new dependent variables

$$\boldsymbol{\psi}^{\pm}(\tau) \equiv \boldsymbol{I}^{+}(\tau) \pm \boldsymbol{I}^{-}(\tau), \tag{3}$$

one arrives at

$$\begin{aligned} &\frac{d^2\boldsymbol{\psi}^{+}(\tau)}{d\tau^2} + \boldsymbol{A}\boldsymbol{\psi}^{+}(\tau) = 0 \\ &\boldsymbol{\psi}^{-}(\tau) = -\left(\boldsymbol{\alpha} + \boldsymbol{\beta}\right)^{-1} \frac{d\boldsymbol{\psi}^{+}(\tau)}{d\tau}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\boldsymbol{\alpha} \equiv \boldsymbol{M}^{-1}\left(\boldsymbol{I}_N - \boldsymbol{C}^{++}\right) \\ &\boldsymbol{\beta} \equiv \boldsymbol{M}^{-1}\boldsymbol{C}^{+-} \\ &\boldsymbol{A} \equiv \left(\boldsymbol{\alpha} + \boldsymbol{\beta}\right)\left(\boldsymbol{\alpha} - \boldsymbol{\beta}\right). \end{aligned}$$

In addition, because of rotationally invariant scattering,

$$\boldsymbol{C}^{++} = \boldsymbol{C}^{--}, \boldsymbol{C}^{+-} = \boldsymbol{C}^{-+}.$$

Through the diagonalization of the Jacobian matrix $\boldsymbol{A}$, one finds the solution

$$\begin{aligned}
\boldsymbol{\psi}^{+}(\tau) &\equiv \boldsymbol{H}^{+}(\tau)\boldsymbol{\psi}^{+}(\tau_0) + \boldsymbol{H}^{-}(\tau)\boldsymbol{\psi}^{+}(0) \\
\boldsymbol{\psi}^{-}(\tau) &= -(\boldsymbol{\alpha}+\boldsymbol{\beta})^{-1}\left[\frac{d\boldsymbol{H}^{+}(\tau)}{d\tau}\boldsymbol{\psi}^{+}(\tau_0) + \frac{d\boldsymbol{H}^{-}(\tau)}{d\tau}\boldsymbol{\psi}^{+}(0)\right],
\end{aligned} \tag{5}$$

where the matrix functions $\boldsymbol{H}^{\pm}$ are

$$\begin{aligned}
\boldsymbol{H}^{+}(\tau) &\equiv \boldsymbol{T}\left[diag\left\{\frac{sinh(\lambda_k \tau)}{sinh(\lambda_k \tau_0)}\right\}\right]\boldsymbol{T}^{-1} \\
\boldsymbol{H}^{-}(\tau) &\equiv \boldsymbol{T}\left[diag\left\{\frac{sinh(\lambda_k(\tau_0-\tau))}{sinh(\lambda_k \tau_0)}\right\}\right]\boldsymbol{T}^{-1}.
\end{aligned}$$

Note that $\boldsymbol{T}$ is a matrix of columns of eigenvectors of $\boldsymbol{A}$ and λ_k, k=1,…,N are the eigenvalues. It is the particular choice of the hyperbolic sine function (*sinh*), rather than a simple exponential to be the solution of the homogeneous equations [Eqs(4)] that provides the necessary stability. Also, note that $\boldsymbol{\psi}^{+}(0)$ and $\boldsymbol{\psi}^{+}(\tau_0)$ are not known.

For the case of a known entering intensity $\boldsymbol{I}^{+}(0)$ at the surface $\tau = 0$ and no entering photons at the surface τ_0, $\boldsymbol{I}^{-}(\tau_0) = 0$, since from Eq(3)

$$\boldsymbol{I}^{\pm}(\tau) = \frac{1}{2}\left[\boldsymbol{\psi}^{+}(\tau) \pm \boldsymbol{\psi}^{-}(\tau)\right], \tag{6}$$

the following algebraic equation:

$$\begin{bmatrix}
\boldsymbol{I}_N - (\boldsymbol{\alpha}+\boldsymbol{\beta})^{-1}\left.\frac{d\boldsymbol{H}^{-}(\tau)}{d\tau}\right|_0 & (\boldsymbol{\alpha}+\boldsymbol{\beta})^{-1}\left.\frac{d\boldsymbol{H}^{+}(\tau)}{d\tau}\right|_0 \\
(\boldsymbol{\alpha}+\boldsymbol{\beta})^{-1}\left.\frac{d\boldsymbol{H}^{-}(\tau)}{d\tau}\right|_{\tau_0} & \boldsymbol{I}_N - (\boldsymbol{\alpha}+\boldsymbol{\beta})^{-1}\left.\frac{d\boldsymbol{H}^{+}(\tau)}{d\tau}\right|_{\tau_0}
\end{bmatrix}
\begin{bmatrix} \boldsymbol{\psi}^{+}(0) \\ \boldsymbol{\psi}^{+}(\tau_0) \end{bmatrix}
=
\begin{bmatrix} 2\boldsymbol{I}^{+}(0) \\ \boldsymbol{0} \end{bmatrix} \tag{7}$$

results for $\left[\boldsymbol{\psi}^{+}(0) \quad \boldsymbol{\psi}^{+}(\tau_0)\right]^T$. This then completes the solution from Eqs(3-7) to give

$$\boldsymbol{I}^{\pm}(\tau)=\frac{1}{2}\left\{\left[\begin{bmatrix}\boldsymbol{H}^{+}(\tau)\mp \\ \mp(\boldsymbol{\alpha}+\boldsymbol{\beta})^{-1}\dfrac{d\boldsymbol{H}^{+}(\tau)}{d\tau}\end{bmatrix}\boldsymbol{\psi}^{+}(\tau_0)\pm\left[\begin{bmatrix}\boldsymbol{H}^{-}(\tau)\mp \\ \mp(\boldsymbol{\alpha}+\boldsymbol{\beta})^{-1}\dfrac{d\boldsymbol{H}^{-}(\tau)}{d\tau}\end{bmatrix}\boldsymbol{\psi}^{-}(0)\right]\right]\right\}. \tag{8}$$

From Eq(8), the reflectance and transmittance follow as

$$R_f \equiv 2\int_0^1 d\mu\mu I(0,\mu) \cong 2\sum_{m=1}^{N}\alpha_m\mu_m I_m(0),\; T_n \equiv 2\int_0^1 d\mu\mu I(\tau_0,-\mu) \cong 2\sum_{m=1}^{N}\alpha_m\mu_m I_m(\tau_0) \tag{9}$$

Numerically, the solution is only approximate since it depends on the orders of the angular discretization N and scattering approximation L. We next describe a method to improve the approximations for the reflectance and transmittance continuously.

Consider the Cloud C1 scattering kernel [5], where L =299—one of the more difficult problems in radiative transfer. Say, we are interested in the extremely accurate determination of reflectance and transmittance in a medium of width 1 mean free path. The scattering albedo is ω = 0.999 and an isotropic intensity enters at $\tau = 0$. Our motivation, in this case, is to provide a high fidelity—nearly nine significant figure— benchmark for methods comparison purposes.

To create a sequence of reflectances and transmittances, the evaluation of Eqs(9) proceeds from angular discretizations N = 2 to 100 by increments of 2. In addition, we tag the scattering order to the angular discretization by letting L be 15+N. In this way, the sequence of reflectances and transmittances are leveraged to improve the approximation. Our measure of accuracy will be convergence of the sequences to their respective limits. It is well known that convergence acceleration tools, such as the Wynn-epsilon (*W-e*) extrapolation [6], accelerate sequence convergence. We will now apply this accelerator.

Table 1 shows the convergence of the original (*Ori*) and *W-e* (shaded) sequences. As observed, the *W-e* accelerated sequence converges to one unit in the 8th place at N about 68; whereas, the original has converged only to six places. Actual convergence of the original to 8-places occurs at N about 110 giving a convergence savings of nearly 40 using *W-e*. When the calculation is performed for the original L of 299 at each N, *W-e* is not effective and the computation time increases indicating the suitability of the strategy used. Table 2 is for a large slab of 64 mean free paths, again indicating accelerated high accuracy.

In conclusion, a new solution to the radiative transfer equation has been described and demonstrated. The solution features an alternative representation of the homogeneous solution to overcome the inherent stiffness. Convergence acceleration to achieve extreme accuracy of 8-

places for reflectance and transmittance is shown. While, in general, 6-place accuracy is certainly sufficient for any application, benchmarking requires extreme accuracy, which is attainable with the new solution.

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Table 1

($\omega = 0.999$, $\tau_0 = 1$, $L = 299$)

N	*Rf* (*Ori*)	*Tn*(*W-e*)	*Rf*(*Ori*)	*Tn*(*W-e*)
10	1.32853262E-01	3.44723794E-01	8.65204197E-01	4.14428111E+00
12	1.33185954E-01	1.33185818E-01	8.64817367E-01	8.64817307E-01
14	1.33185942E-01	1.33185942E-01	8.64817380E-01	8.64817380E-01
16	1.33185145E-01	1.33185834E-01	8.64818177E-01	8.64817300E-01
18	1.33184826E-01	1.33184612E-01	8.64818496E-01	8.64818708E-01
20	1.33184468E-01	1.33183263E-01	8.64818854E-01	8.64819883E-01
22	1.33184204E-01	1.33182992E-01	8.64819118E-01	8.64820330E-01
24	1.33184004E-01	1.33183160E-01	8.64819317E-01	8.64820108E-01
26	1.33183832E-01	1.33183213E-01	8.64819489E-01	8.64820109E-01
28	1.33183693E-01	1.33183693E-01	8.64819629E-01	8.64819629E-01
30	1.33183581E-01	1.33183146E-01	8.64819740E-01	8.64820175E-01
32	1.33183490E-01	1.33183081E-01	8.64819831E-01	8.64820240E-01
34	1.33183415E-01	1.33183036E-01	8.64819906E-01	8.64820285E-01
36	1.33183353E-01	1.33183053E-01	8.64819968E-01	8.64820268E-01
38	1.33183301E-01	1.33183054E-01	8.64820020E-01	8.64820267E-01
40	1.33183258E-01	1.33183053E-01	8.64820064E-01	8.64820268E-01
42	1.33183222E-01	1.33183046E-01	8.64820099E-01	8.64820276E-01
44	1.33183192E-01	1.33183038E-01	8.64820129E-01	8.64820283E-01
46	1.33183167E-01	1.33183043E-01	8.64820154E-01	8.64820278E-01
48	1.33183146E-01	1.33183044E-01	8.64820175E-01	8.64820277E-01
50	1.33183129E-01	1.33183043E-01	8.64820193E-01	8.64820278E-01
52	1.33183114E-01	1.33183049E-01	8.64820208E-01	8.64820272E-01
54	1.33183101E-01	1.33183253E-01	8.64820220E-01	8.64820069E-01
56	1.33183090E-01	1.33183028E-01	8.64820231E-01	8.64820293E-01
58	1.33183081E-01	1.33183028E-01	8.64820240E-01	8.64820293E-01
60	1.33183073E-01	1.33183028E-01	8.64820248E-01	8.64820293E-01
62	1.33183067E-01	1.33183028E-01	8.64820254E-01	8.64820293E-01
64	1.33183061E-01	1.33183028E-01	8.64820260E-01	8.64820293E-01
66	1.33183056E-01	1.33183028E-01	8.64820265E-01	8.64820294E-01
68	1.33183052E-01	1.33183028E-01	8.64820269E-01	8.64820294E-01

Table 2

($\omega = 0.999,\ \tau_0 = 64,\ L = 299$)

N	***Rf* (*Ori*)**	***Tn*(*W-e*)**	***Rf*(*Ori*)**	***Tn*(*W-e*)**
10	8.10977813E-01	2.07053054E+01	8.92602815E-02	-6.18778924E-01
12	8.13202083E-01	8.13202285E-01	8.21041880E-02	8.21413746E-02
14	8.13202372E-01	8.13202372E-01	8.21038814E-02	8.21040798E-02
16	8.13202354E-01	8.13202357E-01	8.21038946E-02	8.21038946E-02
18	8.13202345E-01	8.13202338E-01	8.21039007E-02	8.21039005E-02
20	8.13202337E-01	8.13202237E-01	8.21039068E-02	8.21032943E-02
22	8.13202330E-01	8.13202304E-01	8.21039113E-02	8.21039240E-02
24	8.13202325E-01	8.13202313E-01	8.21039147E-02	8.21039243E-02
26	8.13202321E-01	8.13202308E-01	8.21039175E-02	8.21039239E-02
28	8.13202318E-01	8.13202318E-01	8.21039198E-02	8.21039198E-02
30	8.13202316E-01	8.13202306E-01	8.21039217E-02	8.21039287E-02
32	8.13202314E-01	8.13202304E-01	8.21039231E-02	8.21039297E-02
34	8.13202312E-01	8.13202303E-01	8.21039244E-02	8.21039308E-02
36	8.13202310E-01	8.13202303E-01	8.21039254E-02	8.21039303E-02
38	8.13202309E-01	8.13202303E-01	8.21039262E-02	8.21039303E-02
40	8.13202308E-01	8.13202303E-01	8.21039270E-02	8.21039303E-02
42	8.13202307E-01	8.13202303E-01	8.21039275E-02	8.21039305E-02
44	8.13202307E-01	8.13202303E-01	8.21039280E-02	8.21039306E-02
46	8.13202306E-01	8.13202303E-01	8.21039284E-02	8.21039306E-02
48	8.13202306E-01	8.13202303E-01	8.21039288E-02	8.21039305E-02
50	8.13202305E-01	8.13202303E-01	8.21039291E-02	8.21039305E-02
52	8.13202305E-01	8.13202303E-01	8.21039293E-02	8.21039304E-02
54	8.13202305E-01	8.13202302E-01	8.21039295E-02	8.21039318E-02
56	8.13202304E-01	8.13202303E-01	8.21039297E-02	8.21039307E-02
58	8.13202304E-01	8.13202303E-01	8.21039299E-02	8.21039308E-02
60	8.13202304E-01	8.13202303E-01	8.21039300E-02	8.21039308E-02
62	8.13202304E-01	8.13202303E-01	8.21039301E-02	8.21039307E-02
64	8.13202304E-01	8.13202303E-01	8.21039302E-02	8.21039308E-02
66	8.13202303E-01	8.13202303E-01	8.21039303E-02	8.21039308E-02
68	8.13202303E-01	8.13202303E-01	8.21039304E-02	8.21039308E-02
70	8.13202303E-01	8.13202303E-01	8.21039304E-02	8.21039308E-02