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1. Introduction

The subject of *Dynamical Systems* usually begins with the equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x}\right), \tag{1.1}$$

which is grounded on the Newtonian-Galilean reference frame of an absolute time t and an independent set of spatial variables $\mathbf{x}$. It was based on the above d/dt that Schrödinger presented his non-relativistic quantum wave equation; then Dirac incorporated Einstein's special relativity to arrive at a Lorentz-invariant wave equation. Since then there have been three distinct domains of studies in this regard: the (classical) dynamical systems commonly associated with ordinary differential equations, relativistic physics in the domain of relativity and quantum mechanics, and a developing field going by the generic title of relativistic dynamics. This note by extending the above formulation (1.1) into the domain of special relativity seeks to connect the above disjointed fields of studies. In the next Section 2, we will show how Equation (1.1) can be restored in relativistic dynamics, and in Section 3, we will make a summary remark to suggest possible future studies.

2. Analysis

Let observer O_X be given, with proper time ("wrist watch") t_0 and a 1-dimensional spatial coordinates $\{x_a\} \subset \mathbf{R}_X$. Suppose that O_X observes another observer O_Y moving uniformly along $\mathbf{R}_X$ such that (1) at a spatial coordinate x_1, O_X *could* read O_Y's wrist watch showing t_1, and (2) at a spatial coordinate $x_2 > x_1$, O_X *could* read O_Y's wrist watch showing $t_2 > t_1$. Then O_X would define O_Y's velocity to be

$$v = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} = \frac{dx}{dt} > 0. \tag{1.2}$$

We note that $t \neq t_0$. Symmetrically O_Y observes O_X with velocity

$$\frac{dy}{dt} \equiv -\frac{dx}{dt}. \tag{1.3}$$

Furthermore,

$$L\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \tau \\ y \end{pmatrix}, \textit{ where } L \equiv \textit{ the Lorentz transformation}$$

$$:= \left(1-\left(\frac{v}{c}\right)^2\right)^{-\frac{1}{2}} \begin{pmatrix} 1 & -\frac{v}{c^2} \\ -v & 1 \end{pmatrix}. \tag{1.4}$$

Now consider an emission of light at $t_0 = 0 = \tau_0$ in the direction of v; then $\forall\ t_0,\ \tau_0 > 0$, O_X observes the light traveling at $(t_0,\ t_0\, c)$ and O_Y observes the light traveling at $(\tau_0,\ \tau_0\, c)$. Moreover,

$$\begin{pmatrix} \tau_0 \\ \tau_0 c \end{pmatrix} = L\begin{pmatrix} t_0 \\ t_0 c \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\frac{v}{c^2} \\ -v & 1 \end{pmatrix}\begin{pmatrix} t_0 \\ t_0 c \end{pmatrix} = \gamma \begin{pmatrix} t_0 - t_0 \cdot \frac{v}{c} \\ -vt_0 + t_0 c \end{pmatrix}$$

$$= \gamma\left(1-\frac{v}{c}\right)\begin{pmatrix} t_0 \\ t_0 c \end{pmatrix}, \textit{ where } \gamma \equiv \left(1-\left(\frac{v}{c}\right)^2\right)^{-\frac{1}{2}}. \tag{1.5}$$

I.e.,

$$\gamma\left(1-\frac{v}{c}\right) = \lambda < 1 \tag{1.6}$$

is an eigenvalue of L and

$$\lambda t_0 = \tau_0. \tag{1.7}$$

Returning to observer O_X, we <u>require O_X measure any dynamical variables in terms of his/her proper time t_0</u>, i.e.,

$$\left(t(t_0), x_1(t(t_0)), \ldots, x_n(t(t_0))\right) \in \mathbf{R}^{1+n}. \tag{1.8}$$

Define

$$\mathbf{X}(t_0) := \begin{pmatrix} t(t_0) \\ x_1(t(t_0)) \\ \vdots \\ t(t_0) \\ x_n(t(t_0)) \end{pmatrix} \in \mathbf{R}^{2n}. \tag{1.9}$$

Suppose that

$$\frac{d}{dt_0}\mathbf{X}(t_0)=\mathbf{f}\left(\mathbf{X}(t_0)\right)\ \forall\ t_0 \textit{ in a domain of definition.} \tag{1.10}$$

Define

$$\mathbf{L}:=\begin{pmatrix} \Lambda & O & O & O \\ O & \Lambda & O & O \\ O & O & \ddots & O \\ O & O & O & \Lambda \end{pmatrix}_{2n\times 2n}, \tag{1.11}$$

a block diagonal matrix. Then we have

$$\begin{aligned} &\mathbf{X}(t_0)=\mathbf{L}^{-1}\mathbf{Y}(t_0), \\ &\frac{d}{dt_0}\mathbf{X}(t_0)=\frac{\lambda d}{dt_0}\mathbf{L}^{-1}\mathbf{Y}(t_0), \textit{ and} \\ &\frac{d}{dt_0}\mathbf{Y}(t_0)=\lambda^{-1}\mathbf{L}\mathbf{f}\left(\mathbf{L}^{-1}\mathbf{Y}(t_0)\right)\equiv \mathbf{g}\left(\mathbf{Y}(t_0)\right), \\ &\textit{with } \mathbf{g}\equiv \lambda^{-1}\mathbf{L}\mathbf{f}\mathbf{L}^{-1}. \end{aligned} \tag{1.12}$$

That is, the same dynamics when observed by O_X is

$$\frac{d}{dt_0}\mathbf{X}(t_0)=\mathbf{f}\left(\mathbf{X}(t_0)\right), \tag{1.13}$$

while observed by O_Y is

$$\frac{d}{dt_0}\mathbf{Y}(t_0)=\mathbf{g}\left(\mathbf{Y}(t_0)\right). \tag{1.14}$$

In this way we have restored the formulation of

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}) \tag{1.15}$$

in relativistic dynamics.

3. Summary Remark

In this note we have integrated the classical dynamical systems with special relativity by the scheme of using an observer's proper time to measure all dynamical variables, such as positions $\mathbf{x}(t)$ with their associated times t, velocities, accelerations, energies and momenta, and as a result the Lorentz transformation can be directly applied to any of these dynamical variables. Consequently, dynamical systems in different inertial frames of reference can be related as such. Additional pursuits may be a reaffirmation of all the qualitative properties of a dynamical system across all the inertial reference frames, or the general forms of some well-known ordinary and partial differential equations, and the differential geometry of differential equations that is invariant under arbitrary frames of references.

References

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