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The Anti-derivative of ex^2

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The Anti-derivatives of e^{x^2}

INTRODUCTION

e^{x^2} is an interesting function both in theory and in practice. It is primarily in the realm of *normal distribution* where this function finds its recognition. In particular, the function is useful with standardized normal distribution deriving $\varphi(x)$, the pdf of random variable x.

Several successful attempts have been made in the past to approximate the integral of this function. Methods of numerical integration, Taylor Series, McLaren Series, Simpson's 1/3 formula and Gauss are a few excellent examples, however, none is a true anti-derivative solution: "We cannot apply the Fundamental Theorem of Calculus (to integrate this function) since we cannot find a function whose derivative is $\exp(-x^2/2)$, Meyer. [1]
A. W. Goodman maintains, "surly, there must be some function whose derivative is $\exp(x^2)$…". [2]

It is against the background presented above where the research in this paper begins:

FORMULAE

Let Y be the anti-derivative of e^{x^2}, then two formulae will be presented for Y, one with positive exponent and the other with negative exponent.

Ideal results can be achieved with Y for values selected near the origin. However, the purpose of this paper is not to compete with a foregone method of *definite* integration either in accuracy or in efficiency. Nor is the goal to introduce another method of *approximation.* The mission of this paper, however, is to define a true *anti-derivative* formula as follows:

Theorem:
Let y represent the function $e^{x^2} \Rightarrow dy/dx = 2xy$
Let $n \in \tilde{N}$ (the set of natural numbers).
Let $x \in \Re$ (the set of real numbers), then:

$$Y = \int y dx = \sum_{n=1}^{\infty} \frac{(-4)^{n-1}(n-1)!}{(2n-1)!} x^{2n-1} e^{x^2} + C \qquad (1)$$

Proof: We will show $dy/dx \to y$ as $n \to \infty$

$$dy/dx = y + 2x^2y - 2x^2y + 4/3x^4y - 4/3x^4y + \ldots$$
$$+ (-4)^{n-1}(n-1)!\, x^{2n-2}y / (2n-2)!$$
$$+ 2(-4)^{n-1}(n-1)!\, x^{2n}y / (2n-1)! \qquad (2)$$

It can be seen that all intermediate terms are eliminated if the derivative of the product is taken in the same order as shown in (2). The remaining terms are the first term y and the nth term. Following, we will use the ratio test to show the nth term (odd or even) approaches zero as n approaches infinity:

Let $a_n = 2(-4)^{n-1}(n-1)!\, x^{2n} y / (2n-1)!$ (3)
Then $a_{n+1} = 2(-4)^{n}(n)!\, x^{2n+2} y / (2n+1)!$ (4)
$a_{n+1} / a_n = (4)/(3) = -4x^2 / (2n+1)$ (5)

$$\lim_{n \to \infty} (5) = 0 < 1$$

Thus, the ratio test shows that a_n is *absolutely convergent* for every $x \in \Re$

e^{-x^2} is another interesting function as noted earlier. MML authors put it this way, "The most important function in probability is the density function for the standard normal distribution, which is the familiar bell-shaped curve" [3].

The function is $f(x) = e^{-x^2} / (2\pi)^{1/2.}$

To integrate e^{-x^2}, we will use the formula applied in (1) with the exception of alternating signs because the negative exponent is already present:

Theorem: Let y represent the function e^{-x^2}
Let $n \in \tilde{N}$ (the set of natural numbers).
Let $x \in \Re$ (the set of real numbers), then,

$$\int y\, dx = \sum_{n=1}^{\infty} \frac{4^{n-1}(n-1)!}{(2n-1)!} x^{2n-1} e^{-x^2} + C \qquad (6)$$

The proof is analogous to (2) derived earlier, hence, omitted here for the sake of brevity.

EXAMPLE 1: Find the area under the curve $y = e^{x^2}$ bounded by the x-axis, the y-axis and x = 3. Use the anti-derivative solution, then compare with the trapezoidal rule, Simpson's rule, and Taylor series.

SOLUTION: Formula (1) will be used first, then tabulated against other methods. See Fig. 1

$$A = \int_0^3 e^{x^2} dx = e^{x^2} (x - 2x^3/3 + 4x^5/15 - 8x^7/105 + \dots) \Big|_0^3$$

$$A = e^9 [3 - 2(3)^3/3 + 4(3)^5/15 - \dots + 4^{32}(3)^{65}(32)! / 65!]$$

$A \approx 1444.508736 \rightarrow n = 33$, anti-derivative
$A \approx 1444.690973 \rightarrow n = 500$, trap
$A \approx 1444.545122 \rightarrow n \approx 500$, Simpson
$A \approx 1444.545123 \rightarrow n = 43$, Taylor

EXAMPLE 2: Find the area bounded by $y = e^{-x^2}$, The x-axis, the y-axis and x = 1 (see Fig. II). First use the indefinite integral solution, then compare with Simpson and trapezoidal rules.

SOLUTION: Formula (6) will be used first, then tabulated against other methods. See Fig. II

$$A = \int_0^1 e^{-x^2} dx = e^{-x^2} (x + 2x^3/3 + 4x^5/15 + \dots)$$

$$A = (1/e)(1 + 2/3 + 4/15 + 8/105 + 16/945 + 32/10395 + 64/135135 + 28/2027025)$$

$A \approx 0.7468211 \rightarrow n = 8$, anti-derivative
$A \approx 0.746211 \rightarrow h = 0.1$, trap
$A \approx 0.746825 \rightarrow h = 0.1$, Simpson

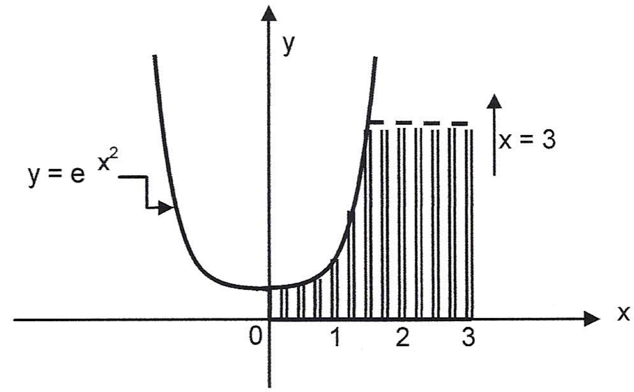

FIG. I

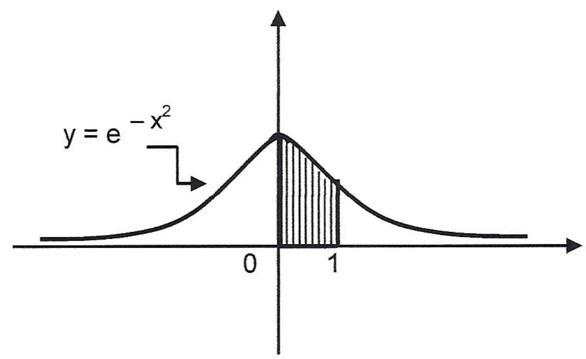

FIG. II

ACKNOWLEDGMENT & REFERENCES

[1] P. L. Meyer, "Introductory Probability And Statistical Applications," 2nd ed. 1970, P. 186

[2] A. W. Goodman, "Analytic Geometry And The Calculus," 9th ed. 1967, P. 281

[3] Lial, Greenwell, Ritchey, "Calculus With Application," 10th ed. 2012, Chapter 7 Post-Test # 18