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# THE MATHEMATICAL BEAUTY OF TRIANGULAR NUMBERS

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# The Mathematical Beauty Of Triangular Numbers

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**Abstract:** The triangular numbers are formed by partial sum of the series  $1+2+3+4+5+6+7+\dots+n$  [2]. In other words, triangular numbers are those counting numbers that can be written as  $T_n = 1+2+3+\dots+n$ . So,

$$T_1 = 1$$

$$T_2 = 1+2=3$$

$$T_3 = 1+2+3=6$$

$$T_4 = 1+2+3+4=10$$

$$T_5 = 1+2+3+4+5=15$$

$$T_6 = 1+2+3+4+5+6=21$$

$$T_7 = 1+2+3+4+5+6+7=28$$

$$T_8 = 1+2+3+4+5+6+7+8=36$$

$$T_9 = 1+2+3+4+5+6+7+8+9=45$$

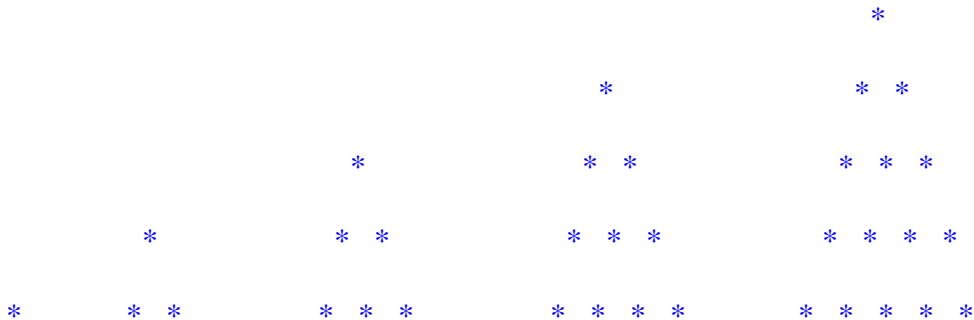
$$T_{10} = 1+2+3+4+5+6+7+8+9+10=55$$

In this paper we investigate some important properties of triangular numbers. Some important results dealing with the mathematical concept of triangular numbers will be proved. We try our best to give short and readable proofs. Most of the results are supplemented with examples.

**Key Words:** Triangular numbers , Perfect square, Pascal Triangles, and perfect numbers.

### 1. Introduction :

The sequence  $1, 3, 6, 10, 15, \dots, n(n+1)/2, \dots$  shows up in many places of mathematics[1]. The Greek called them triangular numbers [1]. The triangular number  $T_n$  is a figurate number that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one as shown below [2].



## The Fascinating Mathematical Beauty Of Triangular Numbers



Mathematicians have been fascinated for many years by the properties and patterns of triangular numbers [2]. We can easily hunt for triangular numbers using the formula:

$$T_n = \frac{n(n+1)}{2}, \quad n > 0.$$

$T_1$	$T_2$	$T_3$	$T_4$	$T_5$	$T_6$	$T_7$	$T_8$	$T_9$	$T_{10}$	$T_{11}$	$T_{12}$	$T_{13}$	$T_{14}$	$T_{15}$	$T_{16}$	$T_{17}$	$T_{18}$	$T_{19}$	$T_{20}$
1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190	210

The first 20 triangular numbers are as follows.

### 2. The Main Results:

**Theorem 1 :** Every triangular number is a binomial coefficient .

Thoerem2. Every T triangular number is an arithmetic progression.

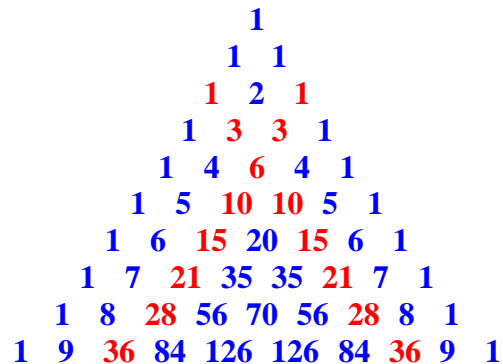
Thoerem 3.

Theorem 4

Theorem 5. Every perfect number is a triangular number.

### Proof without words

Refer to the following Pascal's Triangle [2] and see the red colored numbers.



**Remark1:** Also note that  $T_n = \frac{n(n+1)}{2} = \binom{n+1}{2}$  which is a binomial coefficient for each  $n \geq 1$ .

**Theorem 1:**  $T$  is a triangular number  $\Leftrightarrow 8T+1$  is a perfect square.

**Proof:** (i) ( $\Rightarrow$ ) Assume  $T$  is a triangular number.

$$\text{Let } T = \frac{n(n+1)}{2}, n \text{ a positive integer.}$$

$$\Rightarrow 8T = 8 \frac{n(n+1)}{2}$$

$$\Rightarrow 8T + 1 = 8 \frac{n(n+1)}{2} + 1$$

$$\Rightarrow 8T + 1 = 8 \frac{n(n+1)}{2} + \frac{2}{2}$$

$$\Rightarrow 8T + 1 = \frac{8n^2 + 8n + 2}{2}$$

$$\Rightarrow 8T + 1 = 2 \frac{(4n^2 + 4n + 1)}{2}$$

$$= (2n+1)(2n+1)$$

$$= (2n+1)^2$$

Hence,  $8T+1$  is a perfect square.

(ii) ( $\Leftarrow$ ) Assume  $8T+1$  is a perfect square. Then  $8T+1$  is odd  $\Rightarrow$  for some positive integer  $n$ , we have  $8T+1 = (2n+1)^2 = 4n^2 + 4n + 1 \Rightarrow T = \frac{n(n+1)}{2}$ .

Hence,  $T$  is a triangular number.

By (i) and (ii) the theorem is proved.

**Example 1.** 6 is a triangular number  $\implies 8(6)+1 = 49$  is a perfect square

**Example 2.**  $8(15)+1 = 121$ , a perfect square  $\implies 15$  is a triangular number.

**Corollary 1.**  $T$  is a triangular number  $\iff n = \frac{\sqrt{8T+1}-1}{2}$  is an integer.

**Proof:** The corollary easily follows by **Theorem 1**.

**Theorem 2:** If  $T_m$  and  $T_n$  are triangular numbers, then

$$T_{m+n} = T_m + T_n + mn \quad \text{for } m \text{ and } n \text{ positive integers.}$$

**Proof:**

Note:  $T_m = \frac{m(m+1)}{2}$  &  $T_n = \frac{n(n+1)}{2}$ . Then

$$\begin{aligned} T_m + T_n + mn &= \frac{m(m+1)}{2} + \frac{n(n+1)}{2} + mn \\ &= \frac{m^2 + m + n^2 + n}{2} + mn \\ &= \frac{m^2 + m + n^2 + n + 2mn}{2} = \frac{m^2 + 2mn + n^2 + m + n}{2} \\ &= \frac{(m+n)(m+n) + (m+n)}{2} = \frac{(m+n)[m+n+1]}{2} = T_{m+n} \end{aligned}$$

**Example 3.** Consider  $T_3$  and  $T_4$ . Note that  $T_3 = 6$  and  $T_4 = 10$ . Observe that  $T_{3+4} = T_7 = 28$  and  $T_3 + T_4 + 3(4) = 6 + 10 + 12 = 28$ .

Hence,  $T_{3+4} = T_3 + T_4 + 3(4)$

**Theorem 3:** If  $T_m$  and  $T_n$  are triangular numbers, then

$$T_{mn} = T_m T_n + T_{m-1} T_{n-1}$$

**Proof:** Note:  $T_m = \frac{m(m+1)}{2}$  and  $T_n = \frac{n(n+1)}{2}$ . Then

$$\begin{aligned}
 T_m T_n + T_{m-1} T_{n-1} &= \frac{m(m+1)}{2} \frac{n(n+1)}{2} + \frac{(m-1)m}{2} \frac{(n-1)n}{2} \\
 &= \left( \frac{m^2+m}{2} \right) \left( \frac{n^2+n}{2} \right) + \left( \frac{m^2-m}{2} \right) \left( \frac{n^2-n}{2} \right) \\
 &= \left[ \frac{m^2 n^2 + mn^2 + nm^2 + mn}{4} \right] + \left[ \frac{m^2 n^2 - mn^2 - nm^2 + mn}{4} \right] \\
 &= \frac{2m^2 n^2 + 2mn}{4} = \frac{2mn(mn+1)}{4} = \frac{mn(mn+1)}{2} \\
 &= T_{mn}
 \end{aligned}$$

**Example 4.** Let  $m=6$  and  $n=7$ . Then  $T_6 = 21$  and  $T_7 = 28$ .

By using  $T_n = \frac{n(n+1)}{2}$ , we get  $T_{(6)(7)} = T_{42} = \frac{42(43)}{2} = 903$

We also have  $T_5 T_6 = 15(21) = 315$  and  $T_6 T_7 + T_5 T_6 = 588 + 315 = 903$ .

Hence,  $T_{(6)(7)} = T_{42} = T_6 T_7 + T_5 T_6$

**Lemma 1.** The sum of two consecutive triangular numbers is a perfect square

**Proof:** Let  $T_{n-1}$  and  $T_n$  be any two consecutive triangular numbers, such that

$$T_{n-1} = \frac{(n-1)n}{2} \quad \text{and} \quad T_n = \frac{n(n+1)}{2}$$

Then,

$$T_{n-1} + T_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2}$$

$$= \frac{n^2 - n + n^2 + n}{2} = \frac{2n^2}{2} = n^2$$

which is a perfect square.

**Example 5.** Let  $T_6$  and  $T_7$  be any consecutive triangular numbers. Then  $T_6 + T_7 = 21+28= 49$ , which is a perfect square.

**Lemma 2.**  $1^2 + 2^2 + 3^2 + 4^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$

**Proof.** We can easily prove the lemma using induction.

**Example 6.** Let  $k = 5$ . Then  $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1+4+9+16+25 = 55$ .

Also, we have  $\frac{5(6)(11)}{6} = \mathbf{55}$  and hence  $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \frac{5(6)(11)}{6}$ .

**Theorem 4.** If  $T_k$  be triangular numbers for  $k > 0$ , then we have

$$\sum_{k=1}^n T_k = \frac{n(n+1)(n+2)}{6}$$

**Proof:** To prove the theorem, we apply divide and conquer method by considering two cases:

(1) If  $n$  is even, say  $n = 2k$ , then

$$\begin{aligned} T_1 + T_2 + \dots + T_n &= (T_1 + T_2) + (T_3 + T_4) + \dots + (T_{2k-1} + T_{2k}) \\ &= 2^2 + 4^2 + \dots + (2k)^2 \quad (\text{by Lemma 1}) \\ &= 4(1^2 + 2^2 + \dots + k^2) \\ &= \frac{4k(2k+1)(k+1)}{6} \quad (\text{by Lemma 2}). \\ &= \frac{n(n+1)(n+2)}{6} \quad \text{as } n = 2k \end{aligned}$$

(2) If  $n$  is odd, say  $n=2k+1$ , then

$$\begin{aligned}
 T_1 + T_2 + \dots + T_n &= (T_1 + T_2) + (T_3 + T_4) + \dots + (T_{2k-1} + T_{2k}) + T_{2k+1} \\
 &= 2^2 + 4^2 + \dots + (2k)^2 + \frac{(2k+1)(2k+2)}{2} \quad (\text{by Lemma 1 and definition of } T_k) \\
 &= 4(1^2 + 2^2 + \dots + k^2) + \frac{(2k+1)(2k+2)}{2} \\
 &= \frac{4k(2k+1)(k+1)}{6} + \frac{3(2k+1)(2k+2)}{6} \quad (\text{by Lemma 2.}) \\
 &= \frac{2k(2k+1)(2k+2)}{6} + \frac{3(2k+1)(2k+2)}{6} \\
 &= \frac{(2k+1)(2k+2)(2k+3)}{6} \\
 &= \frac{n(n+1)(n+2)}{6} \quad \text{as } n = 2k+1
 \end{aligned}$$

By (1) and (2) the Theorem is proved.

**Example 7.** Let  $n = 5$ . Then  $\sum_{k=1}^5 T_k = T_1 + T_2 + T_3 + T_4 + T_5 = 1+3+6+10+15= 35$ . We also have,

$$\frac{5(6)(7)}{6} = 35 \quad \text{and hence} \quad \sum_{k=1}^5 T_k = \frac{5(6)(7)}{6}.$$

**Theorem 5** For any natural number  $n$ , the number

$1 + 9 + 9^2 + 9^3 + \dots + 9^n$  is a triangular number.

**Proof:** Let  $T = 1 + 9 + 9^2 + 9^3 + \dots + 9^n$ . Then  $T = \frac{9^{n+1} - 1}{8}$ .

By **Theorem 1**, it suffice to prove that  $8T+1$  is a perfect square. We will apply the divide and conquer method as in **Theorem 4**.

(1) If  $n$  is even, say  $n = 2k$ , then



$$8T+1 = 8 \left( \frac{9^{n+1} - 1}{8} \right) + 1 = 9^{n+1} = 9^{2k+1} = 9(9^{2k}) = (3^{2k+1})^2, \text{ which a perfect square.}$$

(2) If n is odd, say  $n = 2k+1$ , then

$$8T+1 = 8 \left( \frac{9^{n+1} - 1}{8} \right) + 1 = 9^{n+1} = 9^{2k+2} = (9^{k+1})^2, \text{ which a perfect square.}$$

By (1) and (2), the theorem is proved.

**Theorem 6.** If  $T_n$  be triangular numbers for  $n \geq 1$ , then we have

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = 2$$

**Proof:**

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{T_n} \\ &= \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \\ &= 2 \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) \\ &= 2(1) = 2 \end{aligned}$$

**Proposition 2.** The difference of the squares of two consecutive triangular numbers is a cube.

**Proof:** Consider  $T_{n-1} = \frac{(n-1)n}{2}$  and  $T_n = \frac{(n+1)n}{2}$

$$\begin{aligned} \text{Then, } (T_n)^2 - (T_{n-1})^2 &= \left( \frac{n(n+1)}{2} \right)^2 - \left( \frac{(n-1)n}{2} \right)^2 \\ &= \frac{n^4 + 2n^2 + n^2}{4} - \frac{n^4 - 2n^3 + n^2}{4} = \frac{4n^3}{4} = n^3 \end{aligned}$$

**Example 8.** Let  $T_6$  and  $T_7$  be any two consecutive triangular numbers. Then

$$(T_7)^2 - (T_6)^2 = 28^2 - 21^2 = (28+21)(28-21) = (49)(7) = 7^3, \text{ which is a perfect cube.}$$

**Proposition 3:**  $T$  is a triangular is number  $\implies 9T+1$  is a triangular number .

**Proof:** Assume  $T$  is a triangular number.

$$\begin{aligned} \text{Let } T &= \frac{n(n+1)}{2} \\ \implies 9T &= \frac{9n(n+1)}{2} \\ \implies 9T+1 &= \frac{9n(n+1)}{2} + 1 \\ &= \frac{9n(n+1)}{2} + \frac{2}{2} \\ &= \frac{9n(n+1)+2}{2} \\ &= \frac{9n^2+9n+2}{2} \\ &= \frac{(3n+2)(3n+1)}{2} \\ &= \frac{m(m+1)}{2}, \text{ where } m = 3n+1. \end{aligned}$$

Hence,  $9T+1$  is a triangular number.

**Example 9.** Let  $T_8$  be the triangular number. Then  $9T_8+1=45$ , which is a triangular number.

**Proposition 4:**  $n = 2^{k-1} + 2^k + 2^{k+1} + \dots + 2^{2k-2}$  is a triangular number.

**Proof:** Note that  $n = 2^{k-1} + 2^k + \dots + 2^{2k-2} = 2^{k-1}(1 + 2 + 2^2 + \dots + 2^{k-1})$

$$= 2^{k-1}(2^k - 1)$$

$$= \frac{2^k(2^k - 1)}{2}$$

$$= \frac{m(m+1)}{2}, \text{ where } m = 2^k - 1.$$

Hence,  $n$  is a triangular number

**Example 10.** Let  $k=3$ . Then  $n = 2^{3-1} + 2^3 + 2^4 = 28$ , which is a triangular number.

**Proposition 5.**  $n = 1+2+3+4+\dots+(2^k - 1)$  is a triangular number.

**Proof :** Note that  $n = \frac{2^k(2^k - 1)}{2}$

$$= \frac{m(m+1)}{2}, \text{ where } m = 2^k - 1.$$

Hence,  $n$  is a triangular number

**Example 11.** Let  $k = 3$ . Then  $1+2+3+4+5+6+7=28$ , which is a triangular numbers.

**Proposition 6.** Every Perfect number [3] is a triangular number.

**Proof:** Let  $N$  be a perfect number. Then  $N = 2^{k-1}(2^k - 1)$  where  $2^k - 1$  is prime [3]. Note that  $N = 2^{k-1}(2^k - 1) = \frac{2^k(2^k - 1)}{2} = \frac{m(m+1)}{2}$ , where  $m = 2^k - 1$ . Hence  $N$  is a triangular number.

**Proposition 7.** Let  $T_n$  be a triangular number. Then:

$$(1) T_n^2 = T_n + T_{n-1} * T_{n+1}$$

$$(2) T_{n-1}^2 = 2 * T_n * T_{n-1}$$

**Proof:**

$$\begin{aligned} (1) \text{ We have } T_n + T_{n-1} * T_{n+1} &= \frac{n(n+1)}{2} + \frac{(n-1)n}{2} * \frac{(n+1)(n+2)}{2} \\ &= \frac{n(n+1)}{2} + \frac{n^4 + 2n^3 - n^2 - 2n}{4} = \frac{2n^2 + 2n + n^4 + 2n^3 - n^2 - 2n}{4} \\ &= \frac{n^4 + 2n^3 + n^2}{4} \end{aligned}$$

$$= \left( \frac{n(n+1)}{2} \right)^2$$

$$= T_n^2$$

(2) Note that

$$T_{n^2-1} = \frac{(n^2-1)(n^2)}{2}$$

$$= \frac{((n-1)n)(n(n+1))}{2}$$

$$= \frac{2((n-1)n)(n(n+1))}{4}$$

$$= 2 * \frac{(n-1)n}{2} * \frac{n(n+1)}{2}$$

$$= 2 * T_{n-1} * T_n$$

**Conclusion:** We discussed a good number of interesting results on triangular numbers. We strongly believe that this study will be a good encouragement for more deeper investigation into triangular numbers. Triangular numbers are very interesting numbers that are everywhere. These numbers will be the hottest area of concentration in the future for both undergraduate and graduate mathematical research. Both algebraic and figurate aspects of triangular numbers are attractive center of research for mathematicians.

**Open Questions:** The following are lists of open questions that we found.

1. Other than 120 is there an other triangular number which can be expressed as the product of five consecutive numbers ?

Note:  $120 = 1 * 2 * 3 * 4 * 5$

2. Other 9 (refer to Theorem 5) is there any other positive integer n such that

$1+n+n^2+n^3+\dots+n^k$  is a triangular number?

3. Other than 6 is there any other positive integer N such that N, 11N, 111N are all triangular numbers? Note that 6, 66, 666 are triangular numbers.



## References.

1. Jones K, Parker S, and Lemma M: The Mathematical Magic of Perfect Numbers: GaJSci 66(3): 97-106, 2008
2. Gupta S: "Fascinating Triangular Numbers" : p3, 2002
3. Hamburg C: "Triangular Numbers Are Everywhere!": Illinois, Mathematics and Science Academy: p5, 1992.