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PRE-SERVICE TEACHERS UNDERSTANDING OF PROOF AND JUSTIFICATION AT THE ELEMENTARY, MIDDLE, AND HIGH SCHOOL LEVELS

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Pre-service Teachers Understanding of Proof and Justification at the Elementary, Middle, and High School Levels

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Abstract

This paper explores pre-service mathematics teachers' understanding of the proof process at the elementary, middle, and high school levels. The concept of what constitutes proof and justification at the K-12 level were examined in a college geometry course designed for future elementary school, middle school, and high school teachers. Although a variety of geometry theorems were discussed in the course, this paper focuses on the Pythagorean Theorem and its converse. Students developed methods to help elementary, middle, or high school students further their understanding and justification of the Pythagorean Theorem and its converse.

Introduction

Proof is considered an important instrument in the mathematics learning process (Knuth, 2002) as it provides a foundation for meaningful learning. Tall and Mejia-Ramos (2006) noted that proof demonstrates the correctness or incorrectness of mathematics statements and is used in constructing mathematical knowledge. The *Reasoning and Proof* Standard outlined by the National Council of Teachers of Mathematics (NCTM) in <u>Principles and Standards for School Mathematics</u> (2000) is considered one of the most fundamental aspects of mathematics (NCTM, 2000). NCTM (2000) indicates that proof is an important component of mathematics teaching for every age group. It is thus important to provide opportunities for both K-12 students and college-level students that will build their reasoning skills and aid in their understanding of the proof process. Students should be able to test mathematical arguments using direct and indirect reasoning, provide counterexamples to show a mathematical statement is incorrect, and express mathematical expressions using symbolic language (Aylar, 2014). The function of proof has been addressed by many authors (e.g., de Villiers, 1999; Hanna, 1990, 2001; Hersh, 1993) and their work suggests there are various roles proof plays, including: (1) to verify that a statement is true, (2) to explain why a statement is true, (3) to communicate mathematical knowledge, (4) to discover or create new mathematics, or (5) to systematize statements into an axiomatic system. These roles suggest that the concept of proof or justification encompasses convincing, explaining, and understanding (Huang, 2005).

The notion of proof or justification is quite different at the elementary school, middle school, and high school levels. As noted by Jones (1997), in the early grade levels students can be taught to recognize simple patterns and make predictions about them, and ask questions such as, “What would happen if …?”. Justification of mathematical statements in the early grades may constitute nothing more than mathematical verification. At the middle school level, students make conjectures, make and test generalizations, and learn the difference between mathematical explanation and experimental evidence (Jones, 1997). According to the Common Core State Standards for School Mathematics (CCSSI, 2010), by the time students complete the eighth grade they should be able to explain a proof of the Pythagorean Theorem and its converse. In high school, students are expected to extend their mathematical reasoning into understanding and use more rigorous arguments, leading to notions of proof (Jones, 1997). High school students should begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs (CCSSI, 2010). If students in K-12 education are expected to prove or justify mathematical statements, pre-service teachers should have an understanding of what constitutes proof at the elementary school level, middle school level, and high school level.

Although college-level mathematics students may have many experiences testing and justifying mathematical statements, those who are pre-service teachers may not have experiences that allow them to learn what constitutes proof at the elementary school level, middle school level, and high school level. Providing an opportunity for pre-service mathematics teachers to explore what it means to prove and justify mathematical statements at the various levels was the motivation for this research. This paper explores pre-service mathematics teachers' understanding of the proof process at the elementary, middle, and high school levels.

Method

The concept of what constitutes proof and justification at the K-12 level were examined in a one-semester college geometry course designed for future elementary school (grades K-5), middle school (grades 6-8), and high school (grades 9-12) teachers. The study took place at a medium-size public university located in the Upper Midwest region of the United States. There were 18 students in the course; 9 majoring in secondary mathematics education, 6 majoring in elementary mathematics education, and 3 non-teaching mathematics majors. Students were divided into three groups of six students each by the instructor according to the students' interest in teaching elementary school, middle school, high school, or desiring a non-teaching career. Two of the secondary mathematics education majors expressed interest in teaching middle school and one elementary mathematics education major expressed interest in teaching middle school. So, each of the three groups had one student with an interest in teaching at the middle school level. Each group also had one non-teaching mathematics major. Table 1 shows the breakdown of the groups according to interest in the K-12 level they wanted to teach or based on no interest in teaching at the K-12 level.

	Elementary School	Middle School	High School	Non-Teaching
Group 1	2	1	2	1
Group 2	1	1	3	1
Group 3	2	1	2	1

Table 1. Breakdown of groups of students based on teaching interest.

Although many geometry theorems were discussed throughout the semester, for this study the students were asked to focus on proof and justification of the Pythagorean Theorem and its converse. This theorem was chosen because it is a commonly used theorem at the middle school and high school levels, and the students in the course were very familiar with this theorem. As preparation for exploring proof at the various levels, students completed an assignment in which they had to provide a visual proof of the Pythagorean Theorem with explanation, such an algebraic proof. The goal of the assignment used in this study was for students to develop activities that would aid in understanding the Pythagorean Theorem and its converse in elementary school, middle school, and high school levels. Students did not have prior experience in working with the converse of the Pythagorean Theorem.

Each group decided to break into two-person teams in which each team focused on a particular level. In group 1, the non-teaching mathematics major teamed up with the student interested in teaching middle school. In group 2, the non-teaching mathematics major teamed up with the student interested in teaching elementary school while one of the students interested in teaching high school teamed up with a student interested in teaching middle school. In group 3, the non-teaching mathematics major teamed up with a student interested in teaching high school and a student interested in teaching high school teamed up with a student interested in teaching middle school. This allowed the two-person teams within each of the three groups to focus on one K-12 level.

Students were given three class periods to work in their groups to research and develop ideas for the activities and then two class periods were devoted to presentation of the activities. Students also worked outside of class on the assignment. The total time given for students to develop the activities prior to presentation was 10 days. The groups were required to submit a document describing the activity that the K-12 students would be doing at each level making sure that the activities addressed the following questions.

- What does it mean to prove or justify the Pythagorean Theorem at each level?
- What about the converse of the Pythagorean Theorem? How can you help students understand the Pythagorean Theorem and its converse?
- If c is the longest side of a triangle, what relationships hold when comparing c^2 to $a^2 + b^2$? What type of triangles do we get?
- What do you expect students to get from doing the activity?

Students received a group grade, but because each team provided a write-up based on the level they researched, revisions to the assignment were allowed based on feedback from the instructor. The instructor initially provided comments with no score, then after students revised their document, a score with additional comments was given and students were allowed to revise their document one last time before a final score for the assignment was recorded.

Results

The approaches the students took to proving or justifying the Pythagorean Theorem and its converse are discussed in the paragraphs that follow according to each K-12 level: Elementary School, Middle School, and High School. In each case, a description of the activities the groups developed are provided.

Elementary School

There was not much variation in the activities developed by the elementary-school-level teams possibly because at the elementary level students do not formally prove the Pythagorean Theorem. Elementary school students identify the differences between an acute triangle, an obtuse triangle, and a right triangle. One way of identifying these differences is by using the squared lengths of the sides of the triangles. Thus, the activities developed by all three teams focused on having students do calculations with specific lengths of sides of triangles given to them. The activity involved providing students with an acute triangle, an obtuse triangle, and a right triangle. One of the teams had students cutting and maneuvering one-inch squares from grid paper to fit into squares off each side of the three triangles while the other two teams had students maneuvering Starbursts© candy to fit into squares off each side of the three triangles (Keller, 2011). Table 2 shows an example of an obtuse triangle using the one-inch squares and examples of an acute triangle and a right triangle using Starbursts© candy.

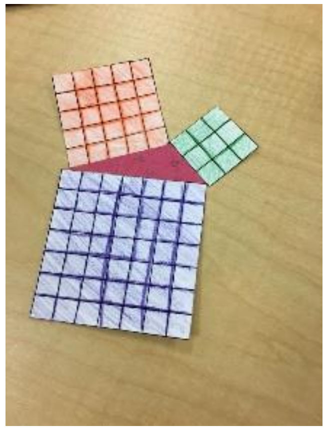
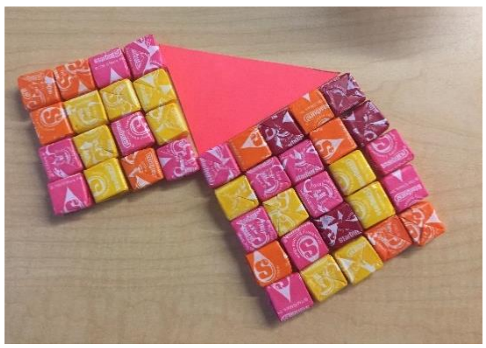
Obtuse Triangle using One-inch Squares		
Acute Triangle using Starbursts© Candy		 
Right Triangle using Starbursts© Candy		

Table 2. Elementary school examples of justifying the Pythagorean Theorem and its converse.

At the elementary level, students will discover that the sum of the areas of the squares constructed off of the shorter sides is equal to the area of the square constructed off of the longest side only in the case of a right triangle. Examples and manipulatives are used to justify the Pythagorean Theorem. Students will be able to generalize that the sum of the areas of the squares constructed off of the shorter sides of the triangle will equal the area of the square constructed off of the longest side of the triangle only if the triangle is a right triangle. If the same number of Starbursts© candy or one-inch squares used to construct squares off of the shorter sides cannot be used to construct a square off of the longest side, the conclusion is that the triangle is not a right triangle.

For example, the students identify that in a right triangle, the sum of the areas of the two smaller squares (9 squares + 16 squares) is the same as the larger square (25 squares). In an acute triangle, the students realize that the sum of the areas of the two smaller squares (16 squares + 25 squares) is greater than the area of the bigger square (36 squares) so there are pieces left over. In an obtuse triangle, the students realize that the sum of the areas of the two smaller squares (9 squares + 25 squares) is less than the area of the bigger square (49 squares) so there are not enough pieces to fill up the area of the square off the longest side of the triangle or there are too many pieces in the area of the square off the longest side of the triangle. The students concluded that, at the elementary school level, these observations by example justify the Pythagorean Theorem and its converse. Thus, at the elementary school level, justification is viewed as mathematical verification rather than a formal mathematical proof.

Middle School

At the middle school level, the students believed that middle school students should be able to apply the Pythagorean Theorem to right triangles to find missing side lengths. In regards to justifying or proving the Pythagorean Theorem, two of the teams used hands-on dissection proofs in combination with explanation and another team used President Garfield's proof with geoboards in combination with an algebraic proof. Figure 1 shows an example layout for Garfield's proof of the Pythagorean Theorem using an interactive geoboard. In the algebraic proof provided by the students, the side lengths of the right triangle were first represented by *a*,

b, and c, where c is the hypotenuse, and then observe that the sum of the areas of the three triangles equals the area of the trapezoid to get $a^2 + b^2 = c^2$.

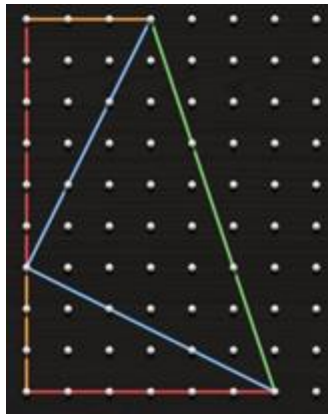


Figure 1. President Garfield's proof of Pythagorean Theorem represented on a geoboard.

For a second activity, this time involving the converse of the Pythagorean Theorem, the team that used Garfield's proof decided that middle school students should experiment with different triangles in order to understand the converse of the Pythagorean Theorem. The middle school students would be given an obtuse, acute, and right triangle cut out of paper, a ruler, and a protractor. The middle school students would then be instructed to measure the lengths of the sides of the triangles and find the measure of the angle between the two shortest sides of the triangles. By measuring the angle between the two shortest sides of the triangle, middle school students would discover which type of triangles satisfied the equation $a^2 + b^2 = c^2$. The observation is that $a^2 + b^2 = c^2$ is true for a right triangle; $a^2 + b^2 < c^2$is true for an obtuse triangle; and $a^2 + b^2 > c^2$ is true for an acute triangle. The middle school students would be expected to make observations and provide a summary of their results.

The two teams that used dissection proofs of the Pythagorean Theorem also used dissection to make observations about the converse of the Pythagorean Theorem. Since both teams used a similar approach, only one is described. Middle school students would be given a worksheet containing dissection pieces and after cutting out the dissection pieces that form the areas of the squares off of legs *a* and *b* they would determine how to fit the dissection pieces in the square off the hypotenuse (see Figure 2). Through this process, middle school students would discover that the Pythagorean Theorem holds only for right triangles.

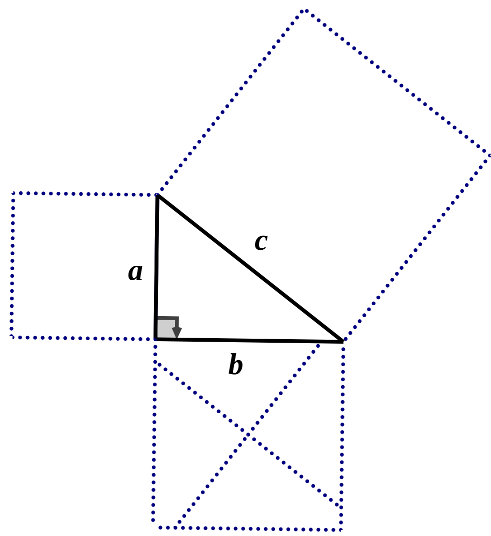


Figure 2. A right triangle with dissection pieces shown on the legs.

Next, the middle school students would try to use the same principles with acute and obtuse triangles. That is, using the dissection pieces from the right triangle, they would attempt to fit them into the square off the longest side of the acute or obtuse triangle (see Figure 3). Note that sides labeled *a* and *b* are the same for all three types of triangles.

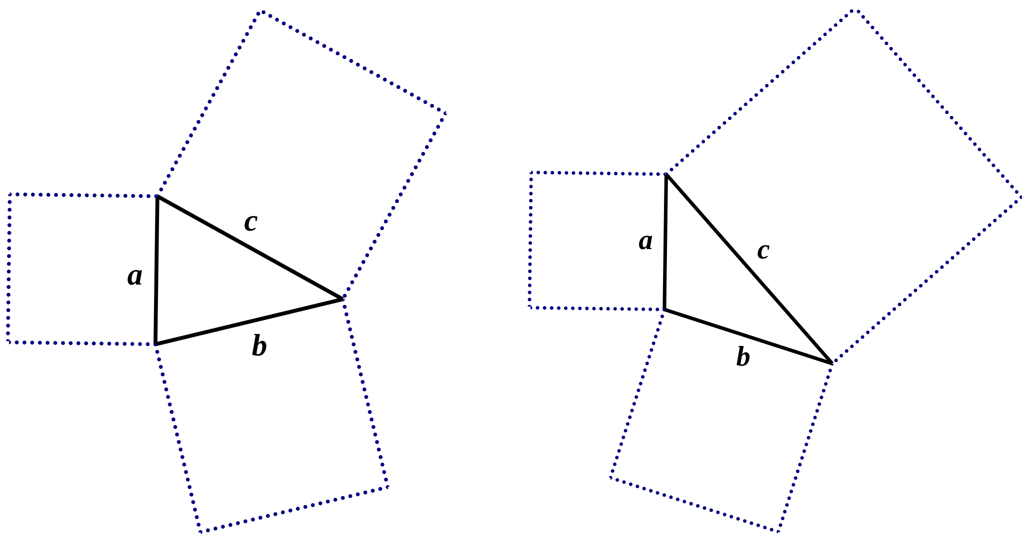


Figure 3. An acute triangle and obtuse triangle with dissection pieces.

To understand how to use the figures in justifying the converse of the Pythagorean Theorem, first, in the case of the acute triangle, the square formed from the dissection pieces is bigger than the square off the longest side, c. Translating these ideas into mathematical expressions and comparing them means that the area of the dissection pieces can be expressed as $a^2 + b^2$ and the area of the square off of c can be expressed as c^2 so the middle school students could conclude that for acute triangles, $a^2 + b^2 > c^2$. For the obtuse triangle, the square formed from the dissection pieces is smaller than the square off the longest side, c. Translating these ideas into mathematical expressions and comparing them means that middle school students should observe the area of the dissection pieces is $a^2 + b^2$ and the area of the square off of side c is c^2 so it follows that for obtuse triangles, $a^2 + b^2 < c^2$.

Visual representations were used to justify the Pythagorean Theorem and its converse at both the middle school level and elementary school level. However, the main difference between justification or proof at the elementary school level and middle school level is that at the elementary school level justification was done by making observations from using specific examples of squares off the sides of the triangles whereas at the middle school level justification was based on not assigning specific values to the side lengths of the triangles and in one case also providing an algebraic proof of the Pythagorean Theorem. The use of visual verification or mathematical proof through algebraic methods coincide with observations made by Huang (2005) in which Hong Kong and Shanghai teachers used the Pythagorean Theorem to introduce the concept of mathematical proof. It should be noted that the two teams using a dissection proof did not include any algebraic proof as did the team that used President Garfield's proof. Instead, these teams required middle school students to provide an explanation of their observations after working with the dissection pieces.

High School

The approaches to proving or justifying the Pythagorean Theorem and its converse were widespread at the high school level. They varied from using a dissection proof similar to the ones described for the middle school level to using straightedge-and-compass constructions (for the converse) to using trigonometry. As the methods varied so much, provided in the paragraphs that follow is a breakdown of how each group approached developing ideas appropriate for high school students might justify the Pythagorean Theorem and its converse.

Group 1

To justify the Pythagorean Theorem, group 1 decided that high school students would be given four congruent right triangles and then be prompted to make a square with the four hypotenuses of the right triangles (side c) creating the sides of the square (see Figure 3). After the high school students created their square they should first explain why the figure created is actually a square and why the inside quadrilateral is also a square. Then they would find the area of the outside square two different ways. First the conventional way, then by adding the areas of the shapes of the inside of the square. This method uses both a visual representation with explanation and an algebraic proof of the Pythagorean Theorem.

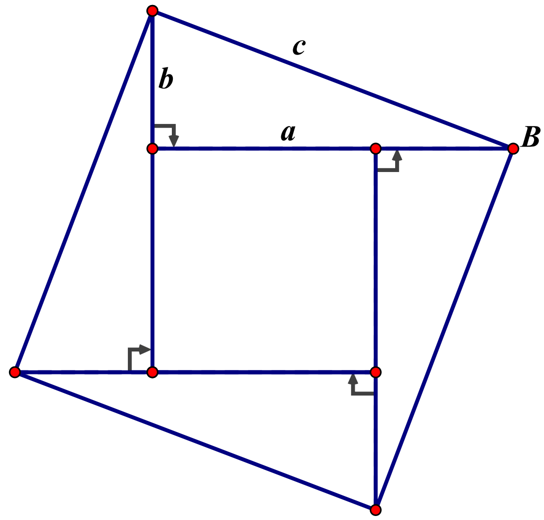


Figure 3. Square made from four congruent right triangles.

For the converse of the Pythagorean Theorem, group 1 created a worksheet with six triangles; two acute triangles, two obtuse triangles, and two right triangles. Each triangle had side lengths given with c the longest side. A table was to be completed with calculations of a^2, b^2, $a^2 + b^2$, c^2, the angle measure between the two shortest sides, an indication of whether the triangle was acute, obtuse, or right, and a column to note <, >, or = in comparing $a^2 + b^2$ to c^2. In addition, there was a question prompting students to summarize their observations from the table. The goal was for high school students to identify the relationship between c^2 and $a^2 + b^2$, where c is the longest side of a triangle. The observation is that when $a^2 + b^2 > c^2$, the triangle is acute and, when $a^2 + b^2 < c^2$, the triangle is obtuse. Thus, the triangle is a right triangle only when $a^2 + b^2 = c^2$.

Group 2

Group 2 decided to use trigonometry to prove the Pythagorean Theorem and its converse. They used the sine and cosine values of a right triangle and the fact that $sin^2\theta + cos^2\theta = 1$ to prove the Pythagorean Theorem. They first used the work of Zimba (2009) to show that $sin^2\theta + cos^2\theta = 1$ from the subtraction formula for the cosine of the difference of angles. By then substituting values for sine and cosine based on their right triangle definitions, the Pythagorean Theorem is established.

For the converse of the Pythagorean Theorem, group 2 constructed an acute triangle and an obtuse triangle using the Geometer's Sketchpad and then used both trigonometry and algebra to determine that when c is the longest side of a triangle, then for an acute triangle $a^2 + b^2 > c^2$, and for an obtuse triangle, $a^2 + b^2 < c^2$. Figure 4 shows the diagram for the acute triangle case. The corresponding proof is provided in figure 5.

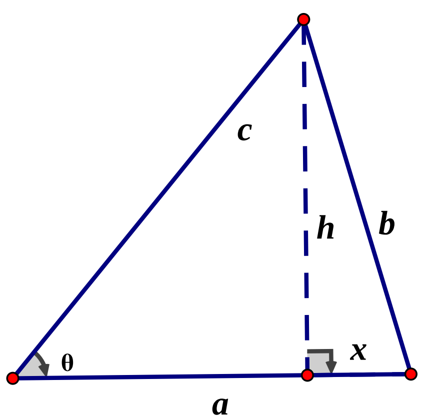


Figure 4. Diagram used for an acute triangle to show $a^2 + b^2 > c^2$.

The students noted that they can observe that $a^2 + b^2 > c^2$ holds true for an acute triangle using the Pythagorean Theorem, but first the height, h, must be constructed and intersect with side a, creating perpendicular segments. The perpendicular segments create 90° angles and right triangles within the acute triangle. In the new right triangle, the trigonometric identity

$sin^2\theta + cos^2\theta = 1$ can now be used to prove $a^2 + b^2 > c^2$. The proof provided by the students is shown below in figure 5.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin\theta = \frac{h}{c} \qquad \cos\theta = \frac{a-x}{c}$$

$$\sin^2\theta = \left(\frac{h}{c}\right)^2 \qquad \cos^2\theta = \left(\frac{(a-x)}{c}\right)^2$$

$$= \frac{h^2}{c^2} \qquad = \frac{(a-x)^2}{c^2}$$

$$\frac{h^2}{c^2} + \frac{(a-x)^2}{c^2} = 1$$

$$\frac{h^2 + x^2 - 2ax + a^2}{c^2} = 1$$

$$h^2 + x^2 - 2ax + a^2 = c^2$$

$$a^2 + b^2 - 2ax = c^2$$

When we remove -2ax, we are left with $a^2 + b^2 > c^2$.

Figure 5. Using the trigonometric identity $sin^2\theta + cos^2\theta = 1$ to prove $a^2 + b^2 > c^2$.

For an obtuse triangle, the students noted that they can observe that $a^2 + b^2 < c^2$ holds true. Figure 6 shows the diagram for the obtuse triangle case.

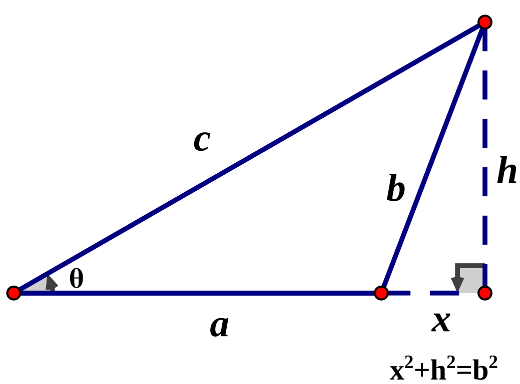


Figure 6. Diagram used for an obtuse triangle to show $a^2 + b^2 < c^2$.

To apply the Pythagorean Theorem, first side *a* must be extended out, creating length x, so that it intersects with the height, *h*, and thus creating a right triangle. In the new right triangle, the trigonometric identity $sin^2\theta + cos^2\theta = 1$ can now be used to prove $a^2 + b^2 < c^2$. The proof provided by the students is shown below in figure 7.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin\theta = \frac{h}{c} \qquad \cos\theta = \frac{a+x}{c}$$

$$\sin^2\theta = \left(\frac{h}{c}\right)^2 \qquad \cos^2\theta = \left(\frac{(a+x)}{c}\right)^2$$

$$= \frac{h^2}{c^2} \qquad = \frac{(a+x)^2}{c^2}$$

$$\frac{h^2}{c^2} + \frac{(a+x)^2}{c^2} = 1$$

$$\frac{h^2 + x^2 + 2ax + a^2}{c^2} = 1$$

$$h^2 + x^2 + 2ax + a^2 = c^2$$

$$a^2 + b^2 + 2ax = c^2$$

When we remove 2ax, we are left with $a^2 + b^2 < c^2$.

Figure 7. Using the trigonometric identity $sin^2\theta + cos^2\theta = 1$ to prove $a^2 + b^2 < c^2$.

Although these proofs were more advanced than what group 1 provided, group 2 thought this method might be appropriate for students in a pre-calculus course in high school. It also shows a level of thinking in the proof process that high school students could achieve.

Group 3

The approach by group 3 to have high school students prove the Pythagorean Theorem was similar to the approach by group 1. The high school students would be given the five cut-out pieces of the figure shown in figure 8 and asked to form a square using the pieces. Algebra would then be used to calculate the area of the outside square and the sum of the areas of the five pieces. Since these areas are equal to each other, then simplifying gives the desired result of $a^2 + b^2 = c^2$.

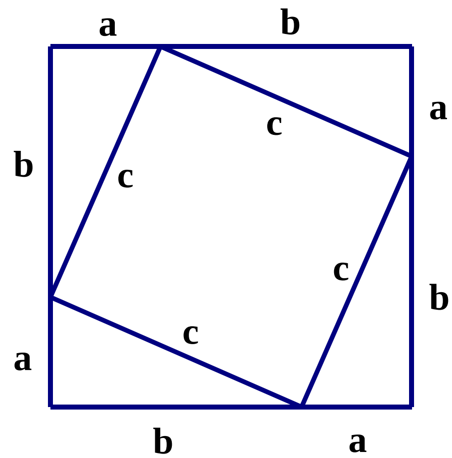


Figure 8. Figure given to high school students to prove Pythagorean Theorem.

To prove the converse of the Pythagorean Theorem, group 3 used the approach outlined in Euclid's Elements (see, for example, Spector (2016)). The high school students would construct triangles using a straightedge and compass, and then complete a formal proof. Group 3 then addressed the following questions, "*If c is the longest side of a triangle, what relationships hold when comparing* c^2 *to* $a^2 + b^2$*? What type of triangles do we get?*" To answer these questions, the high school students would be instructed to construct using a straightedge and compass both an acute triangle and an obtuse triangle with side *c* as the longest side in each case. The high school students would measure the sides of the triangles and be asked to make observations about what relationships occurs between c^2 and $a^2 + b^2$ when <C is an acute angle and when it is an obtuse angle. The observation is that students should see that in the case of the acute triangle $a^2 + b^2 > c^2$ and in the case of the obtuse triangle $a^2 + b^2 < c^2$.

The variation in approaches at the high school level to justification and proof of the Pythagorean Theorem and its converse may be due to the grade level the students had in mind when developing the activities. In the ninth grade, it is possible that students would be expected to use visual representation and algebraic justification similar to that expected in the eighth grade at outlined by CCSSI (2010). However, as students mature in their mathematical thinking, they learn to formalize the proof process as was seen in the activities developed by two of the groups at the high school level.

Summary

In this study, students were asked to develop activities to justify or prove the Pythagorean Theorem and its converse. The goal was to provide experiences for students to learn about proof and justification at the elementary school, middle school, and high school levels. Since reasoning and proof are important components in learning mathematics, it should be a part of K-12 mathematics education. It is thus important for pre-service teachers to learn what constitutes proof at the various levels. As demonstrated by the activities developed by the students in this study, there are multiple approaches to justification and proof. At the elementary school level, the students developed hands-on activities that focused on mathematical verification of the Pythagorean Theorem and its converse as that was what they believed was appropriate for the early grades. At the middle school level, students developed activities that focused on visual proofs of the Pythagorean Theorem using geoboards combined with algebraic verification or allowed for hands-on exploration of the Pythagorean Theorem and its converse using dissection proofs combined with explanation. At the high school level, there was more variety in techniques used to prove the Pythagorean Theorem and its converse, which may have been due to the teams thinking about what might be appropriate for a freshman versus a senior student in high school. The techniques varied from using a dissection proof to using straightedge-and-compass constructions to using trigonometry.

Although there are no clear guidelines as to what constitutes proof at each level, pre-service teachers should have many opportunities to explore its meaning so that they have an understanding of how it can be taught in the schools. The role of proof in K-12 education is an important component in the development of a student's mathematical knowledge and if K-12 students are going to build upon their mathematical reasoning, they need more opportunities to do so, which in turn means that pre-service teachers should be given opportunities to explore the concept of proof in K-12 education.

References

Aylar, E. (2014). A study on the forms of perception of 7th grade students towards the concept of Proof. Journal of Education and Future, 5, 39–56.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/Math/

de Villiers, M. (1999). Rethinking proof with the Geometer's Sketchpad. Emeryville, CA: Key Curriculum Press.

Hanna, G. (1983). Rigorous proof in mathematics education. Toronto: OISE Press.

Hanna, G. (1990). Some pedagogical aspects of proof. Interchange, 21(1), 6–13.

Hersh, R. (1993). Proving is convincing and explaining. Educational Studies in Mathematics, 24, 389–399.

Huang, R. (2005). Verification or proof: Justification of Pythagoras' Theorem in Chinese mathematics classrooms. In Chick, H. L. & Vincent, J. L. (Eds.). Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Vol. 3, pp. 161–168. Melbourne: PME.

Jones, K. (1997). Student-teachers' conceptions of mathematical proof. Mathematics Education Review, 9, 21–32.

Keller, S. K. (2011). Reasoning and proof in the classroom: What's your angle, Pythagoras? A math adventure. Dimensions in Mathematics, 30(2), 21–25.

Knuth, E. J. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. Journal of Mathematics Teachers Education, 5, 61–88.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.

Spector, L. (2016). Plane Geometry: An adventure in language and logic based on Euclid's Elements. Retrieved October 30, 2016, from The Math Pages website: http://www.themathpage.com/aBookI/plane-geometry.htm.

Tall, D. & Mejia-Ramos, J. P. (2006). The long-term cognitive development of different types of reasoning and proof, presented at the Conference on Explanation and Proof in Mathematics: Philosophical and Educational Perspectives, Essen, Germany.

Zimba, J. (2009). On the possibility of trigonometric proofs of the Pythagorean Theorem. Forum Geometricorum, 9, 275–278.