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FEEDBACK LINEARIZATION AND OPTIMAL CONTROL OF ELECTROMAGNETIC BALL SUSPENSION SYSTEM (EMBSS)

NIYIGABA, EMMANUEL ET AL
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DAR-ES-SALAAM
TANZANIA

Mr. Emmanuel Niyigaba
Prof. Santosh Kumar
Department of Mathematics
University of Dar-es-Salaam
Tanzania

Prof. Joyati Debnath
Department of Mathematics and Statistics
Winona State University
Minesota

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Synopsis:

This paper presents the non linear feedback linearization technique and the optimal control of the two models of EMBSS. One of the models has the input as voltage and the other has the input as the current. The two models were analysed and linearized, it was found out that both models are feedback state input-output linearizable. The optimal control was given by the solution of algebraic Riccati equation.

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Emmanuel Niyigaba
Department of Mathematics
University of Dar-es-Salaam
Tanzania, East Africa

Joyati Debnath
Department of Mathematics
and Statistics
Winona State University
Winona, MN, USA

Santosh Kumar
Department of Mathematics
University of Dar-es-Salaam
Tanzania, East Africa

Abstract

This paper presents the non linear feedback linearization technique and the optimal control of the two models of EMBSS. One of the models has the input as voltage and the other has the input as the current. The two models were analysed and linearized, it was found out that both models are feedback state input-output linearizable. The optimal control was given by the solution of algebraic Riccati equation given as follows: $A^TP + PA - PBR^{-1}B^TP + Q = 0$.

Key words

Feedback linearisation, relative degree and Optimal control.

1 Introduction

The EMBSS have many applications especially in transport for high speed (Magnetic Levitation Trains) [Yadav et al., 2013]. The aim of this paper is to present the mathematics that deals with electromagnetic-ball-suspension system especially using ordinary differential equations. We model two different models of electromagnetic-ball-suspension system.

We also need to perform an other technique of linearization called input-output feedback linearization technique [Basile and Marro, 1992]. The feedback linearisation is non linear linearisation technique aiming at designing a controller that linearises completely the system [Joo and Seo, 1996]. It's aim is not to linearise the system which would give the approximate solution around the equilibrium point of the system. Instead we introduce a new control input which linearizes the system in a whole domain of operation. We introducing some notion of Lie derivatives , Lie brackets, relative degree of a non linear control system and the necessary and sufficient condition for a nonlinear control system to be feedback linearised. We performs also the input-state linearisation of the two models; EMBSS with voltage as the input and EMBSS with current as the input. The choice of the input of the control system is not always easy, that's why we need a technique which will help us to find the optimal control law. We shall perform the optimal control technique known as linear quadratic regulator (LQR) [Basile and Marro, 1992].

2 Models Formulation

In this section We formulate the EMBSS models. The EMBSS is a mechanism consisting of electromagnet and a steel ball m as it is shown in Figures:2 and 1. The system functions by regulating the current through electromagnet such that the steel ball of mass m will be suspended at a fixed distance y_0, from the end of electromagnet [Jayawant, 1982]. Our aim is to build two non linear models representing the EMBSS. One model will have the voltage as an input and the other will have the current as an input.

2.1 Model one: EMBSS with Voltage as input

From Figure 1, using Kirchhoffs voltage and current laws one obtains the following equation:

$$\frac{dI}{dt} = -\frac{R}{L}I + \frac{V}{L}. \tag{1}$$

On the other hand, again from Figure: 1, the force F has three components that contribute to it. The first main component is the electromagnetic force $F(y, I)$ which is the attractive force from the electromagnet. It is a function of the position y of the steel ball and the current I through electromagnet. [Suebsomran, 2014] derived this force $F(y, I) = \alpha(\frac{I}{y})^2$, where $\alpha = \frac{1}{4}\mu_0 N^2 A$. Here μ_0 is permeability, N is the number of turns of the coil and A is the cross area section of the electromagnet. The second component is the air resistance which is assumed to be proportional to the velocity of the steel ball, that is to say $F_{air} = -\beta\frac{dy}{dt}$, where β is the proportional coefficient, t is the time and y is the position of the steel ball. The third component is the effect due to the gravitation, that is $F_g = -mg$ where m is the mass and g is the gravitational acceleration; it is assumed to be $9.81m/sec$. Lastly one decides to add another component which serves as a supportive force to the electromagnetic force. This supporting component is chosen to be the force due to a spring placed between the steel ball and the lower end of electromagnet. It is assumed that the system has to obey the Hook's law so that the last component contribution is given by Hook's law; that is $F_s = ky$, where k is spring constant and y is the position attained by the steel ball. Now putting all these force together gives the following:

$$F = \alpha \left(\frac{I}{y}\right)^2 + ky - \beta \frac{dy}{dt} - mg. \qquad (2)$$

Using Newton's laws of motion, one gets the following:

$$\frac{d^2y}{dt^2} = -g + \frac{\alpha}{m}\left(\frac{I}{y}\right)^2 + \frac{k}{m}y - \frac{\beta}{m}\frac{dy}{dt} \qquad (3)$$

The Equations (1) and (3) give us the following system:

$$\begin{cases} \frac{dI}{dt} &= -\frac{R}{L}I + \frac{V}{L}, \\ \frac{d^2y}{dt^2} &= -g + \frac{\alpha}{m}\left(\frac{I}{y}\right)^2 + \frac{k}{m}y - \frac{\beta}{m}\frac{dy}{dt}. \end{cases} \qquad (4)$$

In summary one has the following list is the variable and parameters and their corresponding meaning:

I is the variable current through the electromagnet,
R is the resistor of the circuit,
L is the inductor of the electromagnet,
V is the input voltage,
y is the variable position of the ball,
m is the mass of the ball,
k is the spring constant,
β is the dumping coefficient due to air resistance and any other disturbance,
d is the distance between the ground and electromagnet,
t is the variable time.

2.1.1 The Scope or Domain of the Variables for Model One (Voltage as Input)

In this system, the values of current through the electromagnet is assumed to be positive finite otherwise it would be meaningless to have infinity current. The same applies to the position of the steel ball, in addition the position of the ball can not go beyond the distance between the ground and the end of electromagnet. The derivative of the position can be anything between $-\infty$ and ∞ but finite. The derivative of the current can be anything between $-\infty$ and ∞ but finite. Overall one has the following domain of the variables:

$$\begin{cases} 0 \le I < \infty, \\ 0 < y < d < \infty, \\ -\infty < \frac{dI}{dt} < \infty, \\ -\infty < \frac{dy}{dt} < \infty. \end{cases} \qquad (5)$$

2.1.2 States Space Representation of Model one (Voltage as Input)

Let $I = x_1$, $y = x_2$ and $\frac{dy}{dt} = x_3$ then one has the following system:

$$\begin{cases} \frac{dx_1}{dt} &= -\frac{R}{L}x_1 + \frac{V}{L}, \\ \frac{dx_2}{dt} &= x_3, \\ \frac{dx_3}{dt} &= -g + \frac{\alpha}{m}(\frac{x_1}{x_2})^2 + \frac{k}{m}x_2 - \frac{\beta}{m}x_3, \end{cases} \qquad (6)$$

$$Y(X) = x_2. \qquad (7)$$

The vector X is state of variable vector and it is given by the following equation $X = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$ The output of the system is $Y(X) = x_2$ which is the position of the ball from the electromagnet.

2.2 Model Two: Electromagnetic Ball Suspension System with Current as Input

The current from the source V as indicated by Figure: 2 is passing through the resistor R_1 where the user can change it depending on the output he or she want to get. After the current is regulated the current I is now acting as the input of the system. Then it is divided into two parts one through the capacitor C and other through the resistor R_2 and the electromagnet as shown in Figure: 2. The electromagnet produces attractive force which is capable to suspend the steel ball of mass m. From Figure 2, using Kirchhoff's voltage and current laws we has the following equation:

$$\frac{d^2I_2}{dt^2} = -\frac{R_2}{L}\frac{dI_2}{dt} - \frac{I_2}{LC} + \frac{I}{LC}. \qquad (8)$$

On the other hand, again from Figure:2, the force F has three components that contribute to it. This force is given by the same arguments as it is described in subsection (2.1).
That is $F = \alpha(\frac{I}{y})^2 + ky - \beta\frac{dy}{dt} - mg$ as it is in equation (2). The Newton's laws of motion give the following:

$$\frac{d^2y}{dt^2} = -g + \frac{\alpha}{m}\left(\frac{x_1}{x_3}\right)^2 + \frac{k}{m}y - \frac{\beta}{m}\frac{dy}{dt}. \qquad (9)$$

The Equations (9) and (8) give us the following system:

$$\begin{cases} \frac{d^2I_2}{dt^2} &= -\frac{R_2}{L}\frac{dI_2}{dt} - \frac{I_2}{LC} + \frac{I}{LC}, \\ \frac{d^2y}{dt^2} &= -g + \frac{\alpha}{m}\left(\frac{x_1}{x_3}\right)^2 + \frac{k}{m}y - \frac{\beta}{m}\frac{dy}{dt}. \end{cases} \qquad (10)$$

In summary, one has the following list is the variable and parameters and their corresponding meaning:

I_2 is the variable current through the electromagnet,
R_2 is the resistor of the circuit,
L is the inductor of the electromagnet,
y is the variable position of the ball,
m is the mass of the ball,
k is the spring constant,
β is the dumping coefficient due to air resistance and any other disturbance,
C is the capacitor,
I is the current input of the system,
t is the variable time.

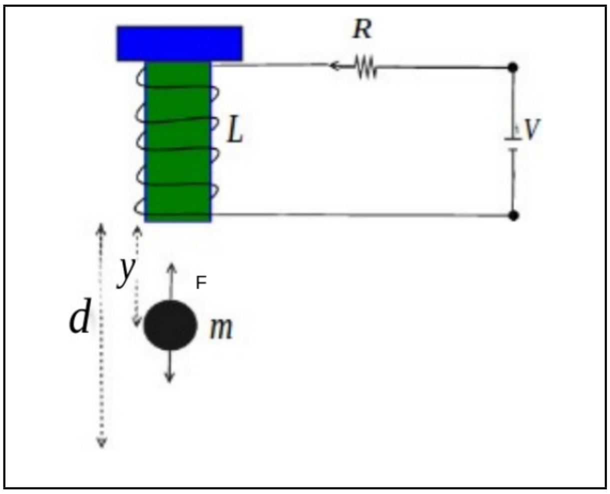

Figure 1. Diagram representing the EMBSS when the input is the voltage.

2.2.1 The Scope or Domain of the Variables for Model Two (Current as Input)

In this system the values of current I_2 through the electromagnet is assumed to be positive finite otherwise it would be meaningless to have infinity current. The same applies to the position of the steel ball, in addition the position of the ball can not go beyond the distance between the ground and the end of electromagnet. The derivative of the position can be anything between $-\infty$ and ∞ but finite. The derivative of the current I_2 can be anything between $-\infty$ and ∞ but finite. Overall one has the following domain of the variables:

$$\begin{cases} 0 \leq I_2 < \infty, \\ 0 < y < d < \infty, \\ -\infty < \frac{dI_2}{dt} < \infty, \\ -\infty < \frac{dy}{dt} < \infty. \end{cases} \tag{11}$$

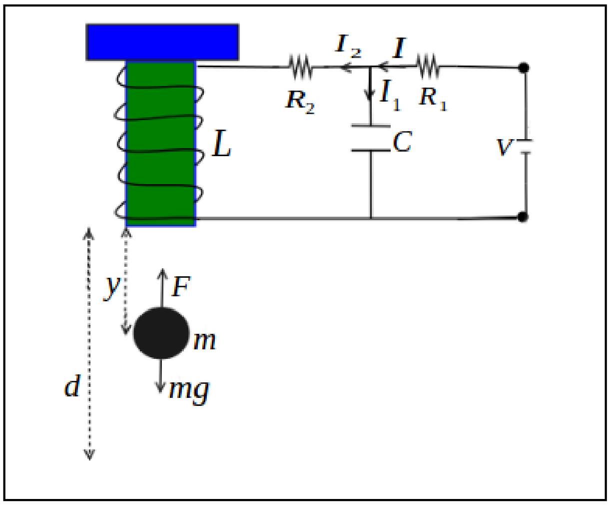

Figure 2. Diagram representing the EMBSS when the input is the current.

2.2.2 States Space Representation of Model Two (Current as Input)

Let $I_2 = x_1$, $\frac{dI_2}{dt} = x_2$, $y = x_3$ and $\frac{dy}{dt} = x_4$, then one has the following system

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -\frac{R_2}{L}x_2 - \frac{x_1}{LC} + \frac{I}{LC}, \\ \frac{dx_3}{dt} = x_4, \\ \frac{dx_4}{dt} = -g + \frac{\alpha}{m}\left(\frac{x_1}{x_3}\right)^2 + \frac{k}{m}x_3 - \frac{\beta}{m}x_4. \end{cases} \tag{12}$$

$$Y(X) = x_3. \tag{13}$$

The vector X is state of variables vector and it is given by the following equation $X = \begin{pmatrix} x_1 \; x_2 \; x_3 \; x_4 \end{pmatrix}^T$. The output of the system is $Y(X) = x_3$ which is the position of the ball from the electromagnet.

2.3 Existence and uniqueness of the solutions of Models one and Two

Definition: A vector- valued function W satisfy a Lipschitz condition in a region D in (X,t)-space if, for some positive constant γ, one has the following inequality $\| \; W(X,t) - W(Y,t) \; \| \leq \gamma \; \| \; X - Y \; \|$, where X and Y are two different space vectors.

Theorem 2.1 (Condition for existence and Uniqueness). *Assume that W(X, t) is continuous and satisfies the Lipschitz condition on the interval $| \; t - t_0 \; | \leq T$ for all X and Y in Domain D. Then the initial value problem*

$$\frac{dX(t)}{dt} = W(X,t), \tag{14}$$

$$X(t_0) = X_0. \tag{15}$$

has a unique solution on the interval $| t - t_0 | \leq T$.

Theorem 2.2 (Existence and Uniqueness of model one). *The system of differential equations given by the system: 6, its solution exist and it is unique for chosen input* V *and initial condition* $X_0 \in D$ *at* $t = 0$, *where* D *is the domain of the variables given by System* (5).

Proof. One needs to show that the vector function $W(X,t)$ is continuous on domain D and satisfies the Lipschitz condition on the same domain D, where $W(X,t)$ and X are given by the following equations:

$$W = \begin{pmatrix} -\frac{R}{L}x_1 + \frac{V}{L} \\ x_3 \\ -g + \frac{\alpha}{m}(\frac{x_1}{x_2})^2 + \frac{k}{m}x_2 - \frac{\beta}{m}x_3 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \ x_2 \ x_3 \end{pmatrix}^T. \tag{16}$$

It is clear that the W is continuous on the domain D. It remains to show that it satisfy the Lipschitz condition; that is for any two vector X and Y and for some positive real number γ, one has the following inequality:

$$\| W(X,t) - W(Y,t) \| \leq \gamma \| X - Y \|. \tag{17}$$

Now let us assume that $Y = \begin{pmatrix} y_1 \ y_2 \ y_3 \end{pmatrix}^T$.

$$\begin{aligned} &\| W(X,t) - W(Y,t) \|^2 \\ &= \left(\frac{R}{L}\right)^2 (x_1 - y_1)^2 + (x_3 - y_3)^2 \\ &\quad + \left(\frac{\alpha}{m}\left(\left(\frac{x_1}{x_2}\right)^2 - \left(\frac{y_1}{y_2}\right)^2\right) + \Upsilon\right)^2, \qquad (18) \\ &= \left(\frac{R}{L}\right)^2 (x_1 - y_1)^2 + (x_3 - y_3)^2 \\ &\quad + \left(\frac{\alpha}{m}\left(\left(\frac{x_1 y_2}{x_2 y_2}\right)^2 - \left(\frac{y_1 x_2}{y_2 x_2}\right)^2\right) + \Upsilon\right)^2 \qquad (19) \end{aligned}$$

In equations (18) and (19), $\Upsilon = \frac{k}{m}(x_2 - y_2) + \frac{\beta}{m}(x_3 - y_3)$. On the other hand one knows that $0 < x_2 < d$ and $0 < y_2 < d$. Also using Equation (1) and replacing I by x_1 it is easy to show that $0 \leq x_1 < \frac{V}{R}$; and hence $0 \leq y_1 < \frac{V}{R}$. From this results one can have the following:

$$(y_1 x_2)^2 < \left(\frac{Vd}{R}\right)^2 \quad \text{and} \quad (y_2 x_1)^2 < \left(\frac{Vd}{R}\right)^2. \tag{20}$$

Put (20) in (19), one has the following:

$$\begin{aligned} &\| W(X,t) - W(Y,t) \|^2 \\ &\leq \left(\frac{R}{L}\right)^2 (x_1 - y_1)^2 + (x_3 - y_3)^2 \\ &\quad + \left(\frac{k}{m}(x_2 - y_2) + \frac{\beta}{m}(x_3 - y_3)\right)^2, \\ &\leq \left(\frac{R}{L}\right)^2 (x_1 - y_1)^2 + (x_3 - y_3)^2 + \frac{k^2}{m^2}(x_2 - y_2)^2 \\ &\quad + \frac{\beta^2}{m^2}(x_3 - y_3)^2 + 2\frac{\beta k}{m^2}(x_2 - y_2)(x_3 - y_3). \qquad (21) \end{aligned}$$

Since $0 < x_2 < d$ and $0 < y_2 < d$, without loosing generality the inequality (21) gives the following:

$$\begin{aligned} &\| W(X,t) - W(Y,t) \|^2 \\ &\leq \left(\frac{R}{L}\right)^2 (x_1 - y_1)^2 + (x_3 - y_3)^2 \\ &\quad + \left(\frac{k}{m}(x_2 - y_2) + \frac{\beta}{m}(x_3 - y_3)\right)^2, \\ &\leq \left(\frac{R}{L}\right)^2 (x_1 - y_1)^2 + (x_3 - y_3)^2 \\ &\quad + \frac{k^2}{m^2}(x_2 - y_2)^2 + \frac{\beta^2}{m^2}(x_3 - y_3)^2 \\ &\quad + 2\frac{\beta k}{m^2}(d - d)(x_3 - y_3), \\ &\leq \left(\frac{R}{L}\right)^2 (x_1 - y_1)^2 + (x_3 - y_3)^2 + \frac{k^2}{m^2}(x_2 - y_2)^2 \\ &\quad + \frac{\beta^2}{m^2}(x_3 - y_3)^2, \\ &\leq \left(\frac{R}{L}\right)^2 (x_1 - y_1)^2 + \frac{k^2}{m^2}(x_2 - y_2)^2 \\ &\quad + \left(1 + \frac{\beta^2}{m^2}\right)(x_3 - y_3)^2. \qquad (22) \end{aligned}$$

Now if we choose $\gamma^2 = \max(\left(\frac{R}{L}\right)^2, \frac{k^2}{m^2}, \left(1 + \frac{\beta^2}{m^2}\right))$ then the inequality (23) yields the following:

$$\begin{aligned} &\| W(X,t) - W(Y,t) \|^2 \\ &\leq (\gamma)^2 (x_1 - y_1)^2 + (\gamma)^2 (x_2 - y_2)^2, \\ &\quad + (\gamma)^2 (x_3 - y_3)^2 \\ &\leq (\gamma)^2 [(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2] \\ &\leq (\gamma)^2 \| X - Y \|^2. \qquad (23) \end{aligned}$$

The inequality (23) gives us the following:

$$\| W(X,t) - W(Y,t) \|^2 \leq (\gamma)^2 \| X - Y \|^2.$$

which yields that

$$\| W(X,t) - W(Y,t) \| \leq \gamma \| X - Y \|.$$

Hence $W(X,t)$ satisfies the Lipschitz condition. Since it is continuous on the domain D and Lipschitz of order one on the domain of variables D, one can conclude that the solution to system 6 exists and it is unique for any chosen initial value problem from D.

Theorem 2.3 (Existence and Uniqueness of Model two). *The system of differential Equations given by the system: 12, its solution exists and it is unique for chosen input I and initial condition $X_0 \in D$ at $t = 0$. Where D is the domain of the variables given by System* (11).

Proof. One needs to show that the vector function $W(X,t)$ is continuous on domain D, and satisfies the Lipschitz condition on the same domain D, where $W(X,t)$ and X are given by the following equations:

$$W = \begin{pmatrix} x_2 \\ -\frac{R_2}{L}x_2 - \frac{1}{LC}x_1 + \frac{I}{LC} \\ x_4 \\ -g + \frac{\alpha}{m}(\frac{x_1}{x_3})^2 + \frac{k}{m}x_3 - \frac{\beta}{m}x_4 \end{pmatrix},$$
$$X = \begin{pmatrix} x_1 \ x_2 \ x_3 \ x_4 \end{pmatrix}^T. \tag{24}$$

It is clear that the W is continuous on the domain D. It remains to show that it satisfy the Lipschitz condition; that is for any two vector X and Y and for some positive real number L, one has the following inequality:

$$\| W(X,t) - W(Y,t) \| \leq \gamma \| X - Y \| \tag{25}$$

Now let assume that $Y = \begin{pmatrix} y_1 \ y_2 \ y_3 \ y_4 \end{pmatrix}^T$.

$$\begin{aligned} &\| W(X,t) - W(Y,t) \|^2 = (x_2 - y_2)^2 \\ &+ \left(\frac{R_2}{L}(y_2 - x_2) + \frac{1}{LC}(x_1 - y_1) \right)^2 + (x_4 - y_4)^2 \\ &\left(\frac{\alpha}{m} \left(\left(\frac{x_1 y_3}{x_3 y_3} \right)^2 - \left(\frac{y_1 x_3}{y_3 x_3} \right)^2 \right) + H \right)^2. \end{aligned} \tag{26}$$

In equation (26), $H = \frac{k}{m}(x_3 - y_3) + \frac{\beta}{m}(y_4 - x_4)$. On the other hand one knows that $0 < x_3 < d$ and $0 < y_3 < d$. Also using Equation 8 and replacing I_2 by x_1 it is easy to show that $0 \leq x_1 < \delta$; and hence $0 \leq y_1 < \delta$ where δ is the maximum value of $x_1(t)$ on $0 \leq t < \infty$. From this results one can have the following:

$$\begin{aligned} &(y_1 x_3)^2 < (\delta d)^2 \quad \text{and} \\ &(y_3 x_1)^2 < (\delta d)^2. \end{aligned} \tag{27}$$

Put (27) in (26), we obtain the following:

$$\begin{aligned} &\| W(X,t) - W(Y,t) \|^2 \qquad (28) \\ &\leq (x_2 - y_2)^2 + \left(\frac{R_2}{L}(y_2 - x_2) + \frac{1}{LC}(x_1 - y_1) \right)^2 \\ &\quad + (x_4 - y_4)^2 + \left(\frac{k}{m}(x_3 - y_3) + \frac{\beta}{m}(y_4 - x_4) \right)^2, \\ &\leq (x_2 - y_2)^2 + \frac{R_2^2}{L^2}(y_2 - x_2)^2 + \frac{1}{L^2 C^2}(x_1 - y_1)^2 \\ &\quad + 2\frac{R_2}{L^2 C}(y_2 - x_2)(x_1 - y_1) \\ &\quad + (x_4 - y_4)^2 + \frac{k^2}{m^2}(x_3 - y_3)^2 + \frac{\beta^2}{m^2}(y_4 - x_4)^2 \\ &\quad + 2\frac{k\beta}{m^2}(x_3 - y_3)(y_4 - x_4). \end{aligned} \tag{29}$$

Since $0 < x_3 < d$, $0 < y_3 < d$, $0 \leq x_1 < \delta$ and $0 \leq y_1 < \delta$ without loosing generality the inequality (29) yields the following:

$$\begin{aligned} &\| W(X,t) - W(Y,t) \|^2 \leq (x_2 - y_2)^2 + \frac{R_2^2}{L^2}(y_2 - x_2)^2 \\ &\quad + \frac{1}{L^2 C^2}(x_1 - y_1)^2 + (x_4 - y_4)^2 \\ &\quad + \frac{k^2}{m^2}(x_3 - y_3)^2 + \frac{\beta^2}{m^2}(y_4 - x_4)^2, \\ &\leq \frac{1}{L^2 C^2}(x_1 - y_1)^2 + (\frac{R_2^2}{L^2} + 1)(x_2 - y_2)^2 \\ &\quad + \frac{k^2}{m^2}(x_3 - y_3)^2 + (\frac{\beta^2}{m^2} + 1)(x_4 - y_4)^2. \end{aligned} \tag{30}$$

Now if we choose

$$\gamma^2 = \max(\frac{1}{L^2 C^2}, (\frac{R_2^2}{L^2} + 1), \frac{k^2}{m^2}, (\frac{\beta^2}{m^2} + 1)),$$

then the inequality (30) yields the following:

$$\begin{aligned} &\| W(X,t) - W(Y,t) \|^2 \leq (\gamma)^2 (x_1 - y_1)^2 + (\gamma)^2 (x_2 - y_2)^2 \\ &\quad + (\gamma)^2 (x_3 - y_3)^2 + (\gamma)^2 (x_4 - y_4)^2, \\ &\leq (\gamma)^2 [(x_1 - y_1)^2 + (x_2 - y_2)^2 \\ &\quad + (x_3 - y_3)^2 + (x_4 - y_4)^2], \\ &\leq (\gamma)^2 \| X - Y \|^2. \end{aligned}$$

This gives us the following:

$$\| W(X,t) - W(Y,t) \|^2 \leq (\gamma)^2 \| X - Y \|^2, \tag{31}$$

which yields that

$$\| W(X,t) - W(Y,t) \| \leq \gamma \| X - Y \|.$$

Hence $W(X,t)$ satisfies the Lipschitz condition. Since it is continuous on the domain D and Lipschitz of order one on the domain of variables D, one conclude that the solution to system 12 exists and it is unique for any chosen initial value problem from D.

2.4 Feedback Linearization

Feedback linearization helps to transform the equations (6) and (12) in a linear one of the following form:

$$\dot{Z} = MZ + NV, \tag{32}$$

where M and N are time invariant matrices and V is the new input. Here V is chosen in such a way that the system is stabilized ie one can chose $V = -KX$ in such a way that the matrix $M - NK$ is Hurwitz or all its eigenvalues has negative real parts [Basile and Marro, 1992].

Definition (Lie derivative): Let $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth scalar function and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field on $\mathbb{R}^n$, then Lie derivative of h with respect to F is a scalar function defined by $L_F h = (\nabla h)F$ [Al Hokayem and Gallestey, 2015].

Definition (Lie bracket): Let F and G be two vector field on $\mathbb{R}^n$.The Lie bracket of F and G is an other vector defined by $[F,G] = (\nabla G)F - (\nabla F)G$ [Al Hokayem and Gallestey, 2015]. It is commonly written as ad_{FG} where ad stand for adjoint. It can be shown that the following chain is holding:

$$\begin{aligned}
ad_{F^0G} &= G,\\
ad_{F^1G} &= [F,G],\\
ad_{F^2G} &= [F, ad_{FG}],\\
&\;\;\vdots\\
ad_{F^iG} &= [F, ad_{F^{i-1}G}].
\end{aligned}$$

Properties of Lie Bracket The Lie bracket satisfy the following properties:

1. Bilinearity: $[\alpha_1 F_1 + \alpha_2 F_2, G] = \alpha_1[F_1,G] + \alpha_2[F_2,G]$,
2. Skey commutativity: $[F,G] = -[G,F]$,
3. Jacobi identity: $L_{[F,G]}h = L_F L_G h - L_G L_F h$.

2.4.1 Relative Degree of a non Linear Control System

In general the non linear control system is represented by the following equations.

$$\frac{dX}{dt} = F(X) + G(X)U, \tag{33}$$
$$Y = h(X).$$

where U is the input of the system and Y is the out put of the system. Let a non linear control system be given by Equation (33), then the relative degree of this system is defined as the number r at the point X_0 such that for $L_G[L_F^k h(X)] = 0$ for all X in the neighbourhood of X_0 for all $k < r-1$ and $L_G[L_F^{r-1}h(X_0)] \neq 0$ [Al Hokayem and Gallestey, 2015].

Theorem 2.4. *The EMBSS given by the system* (6) *has three relative degree at every point* $X = (x_1, x_2, x_3)$ *of the domain of variable* D *where* $x_1 \neq 0$ *and* $x_2 \neq 0$

Proof. The system (6) is given in the form of the system (33) where,

$$F(X) = \begin{pmatrix} -\frac{R}{L}x_1 \\ x_3 \\ -g + \frac{\alpha}{m}(\frac{x_1}{x_2})^2 + \frac{k}{m}x_2 - \frac{\beta}{m}x_3 \end{pmatrix},$$
$$X = \begin{pmatrix} x_1 \; x_2 \; x_3 \end{pmatrix}^T,$$
$$G(X) = \begin{pmatrix} \frac{1}{L} \\ 0 \\ 0 \end{pmatrix}, Y(X) = x_2 \quad \text{and} \quad U(X) = V.$$

Now we have to calculate the derivatives with respect to the independent variable t of the output $Y(X)$ until one gets a value which is different from zero. we need to apply successively the Lie derivatives. This will lead us to the following calculations:

$$\begin{aligned}
L_G L_F^0 Y &= \begin{pmatrix} \frac{\partial Y}{\partial x_1} & \frac{\partial Y}{\partial x_2} & \frac{\partial Y}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{1}{L} \\ 0 \\ 0 \end{pmatrix},\\
&= \begin{pmatrix} 0 \; 1 \; 0 \end{pmatrix} \begin{pmatrix} \frac{1}{L} \\ 0 \\ 0 \end{pmatrix} = 0.
\end{aligned} \tag{34}$$

$$\begin{aligned}
L_G L_F^1 Y &= L_G(L_F Y),\\
&= L_G\left[\begin{pmatrix} 0 \; 1 \; 0 \end{pmatrix} \begin{pmatrix} -\frac{R}{L}x_1 \\ x_3 \\ \Delta \end{pmatrix} \right], \qquad (35)\\
&= L_G(x_3) = \begin{pmatrix} 0 \; 0 \; 1 \end{pmatrix} \begin{pmatrix} \frac{1}{L} \\ 0 \\ 0 \end{pmatrix} = 0. \qquad (36)
\end{aligned}$$

$$\begin{aligned}
L_G L_F^2 Y &= L_G\left[L_F\left(L_F Y\right)\right],\\
&= L_G\left[L_F\left(x_3\right)\right] = L_G[\begin{pmatrix} 0 \; 0 \; 1 \end{pmatrix} F],\\
&= L_G\left(-g + \frac{\alpha}{m}(\frac{x_1}{x_2})^2 + \frac{k}{m}x_2 - \frac{\beta}{m}x_3\right),\\
&= \begin{pmatrix} \frac{2\,\alpha x_1}{m x_2^2} & \frac{k}{m} - \frac{2\,\alpha x_1^2}{m x_2^3} & -\frac{\beta}{m} \end{pmatrix} \begin{pmatrix} \frac{1}{L} \\ 0 \\ 0 \end{pmatrix}, \qquad (37)\\
&= \frac{2\alpha x_1}{m L x_2^2}. \qquad (38)
\end{aligned}$$

where in equation (35), $\Delta = g + \frac{\alpha}{m}(\frac{x_1}{x_2})^2 + \frac{k}{m}x_2 - \frac{\beta}{m}x_3$. From Equation (38), we find out that $L_G L_F^i Y$ fails to be identically zero when $i = 2$ and $x_1 \neq 0$ and $x_2 \neq 0$. Thus the relative degree is given by $r = 2 + 1 = 3$ if $x_1 \neq 0$ and $x_3 \neq 0$.

Theorem 2.5. *The EMBSS given by the system* (12) *has four relative degree at every point* $X = (x_1, x_2, x_3, x_4)$ *of the domain of variable* D *where* $x_1 \neq 0$ *and* $x_3 \neq 0$.

Proof. The system (12) is given in the form of the system (33) where,

$$F(X) = \begin{pmatrix} x_2 \\ -\frac{R_2}{L}x_2 - \frac{x_1}{Lc} \\ x_4 \\ g + \frac{\alpha}{m}(\frac{x_1}{x_3})^2 + \frac{k}{m}x_3 - \frac{\beta}{m}x_4 \end{pmatrix},$$
$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix},$$
$$G(X) = \begin{pmatrix} 0 \\ \frac{1}{LC} \\ 0 \\ 0 \end{pmatrix}, Y(X) = x_3 \quad \text{and} \quad U(X) = I.$$

Now we have to calculate the derivatives with respect to the independent variable t of the output $Y(X)$, until we get a value which is different from zero for some X in the domain of variables. we need to apply successively the Lie derivatives. This will lead us to the following calculations:

$$L_G L_F^0 Y = \begin{pmatrix} \frac{\partial Y}{\partial x_1} & \frac{\partial Y}{\partial x_2} & \frac{\partial Y}{\partial x_3} & \frac{\partial Y}{\partial x_4} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{LC} \\ 0 \\ 0 \end{pmatrix},$$
$$= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{LC} \\ 0 \\ 0 \end{pmatrix} = 0. \tag{39}$$

$$L_G L_F^1 Y = L_G(L_F Y),$$
$$= L_G \left[\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ -\frac{R_2}{L}x_2 - \frac{x_1}{Lc} \\ x_4 \\ \varphi \end{pmatrix} \right] \tag{40}$$
$$= L_G(x_4) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{LC} \\ 0 \\ 0 \end{pmatrix} = 0,$$

where $\varphi = g + \frac{\alpha}{m}(\frac{x_1}{x_3})^2 + \frac{k}{m}x_3 - \frac{\beta}{m}x_4$.

$$L_G L_F^2 Y = L_G\left[L_F\left(L_F Y\right)\right] = L_G\left[L_F\left(x_3\right)\right],$$
$$= L_G[\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} F],$$
$$= L_G\left(-g + \frac{\alpha}{m}(\frac{x_1}{x_3})^2 + \frac{k}{m}x_3 - \frac{\beta}{m}x_4\right)$$
$$= \begin{pmatrix} \frac{2\,\alpha x_1}{m x_3^2} & 0 & \frac{k}{m} - \frac{2\,\alpha x_1^2}{m x_3^3} & -\frac{\beta}{m} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{LC} \\ 0 \\ 0 \end{pmatrix} = 0.$$

$$L_G L_F^3 Y = L_G[L_F(L_F^2 Y)],$$
$$= L_G\left[L_F\left(-g + \frac{\alpha}{m}\left(\frac{x_1}{x_3}\right)^2 + \frac{k}{m}x_3 - \frac{\beta}{m}x_4\right)\right],$$
$$= \begin{pmatrix} A & \frac{2\,\alpha x_1}{m x_3^2} & B & C \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{LC} \\ 0 \\ 0 \end{pmatrix} = \frac{2\,\alpha x_1}{mLC x_3^2}, \tag{41}$$

where A, B and C are given by the following equations

$$A = -\frac{2\left(2\,\alpha m x_1 x_4 + (\alpha\beta x_1 - \alpha m x_2)x_3\right)}{m^2 x_3^3},$$
$$B = -\frac{\beta k x_3^4 - 6\,\alpha m x_1^2 x_4 - 2\left(\alpha\beta x_1^2 - 2\,\alpha m x_1 x_2\right)x_3}{m^2 x_3^4},$$
$$C = -\frac{2\,\alpha m x_1^2 - \left(\beta^2 + km\right)x_3^3}{m^2 x_3^3}.$$

From Equation (41), we find out that $L_G L_F^i Y$ fail to be identically zero when $i = 3$ and $x_1 \neq 0$ and $x_3 \neq 0$. Thus the relative degree is given by $r = 3 + 1 = 4$ if $x_1 \neq 0$ and $x_3 \neq 0$.

Theorem 2.6 ((Frobenius):). ***The Necessary and sufficient condition for input state linearisation.***
The system given by the system (33) *with* $F(X)$ *and* $G(X)$ *being smooth vector fields, is said to be input-state linearisable if and only if there exists a region* D *such that the following conditions hold:*

1. *the vector fields* $\{G, ad_F G, ..., ad_F^{n-1} G\}$ *are linearly independent in D,*
2. *the vector fields* $\{G, ad_F G, ..., ad_F^{n-2} G\}$ *is involutive in D.*

2.4.2 Algorithm for Input-output Linearisation

Find the relative degree of r of the non linear system.
Make r transformations as follows:

$$Z = \Phi(X) = \begin{pmatrix} \varphi_1(X) \\ \varphi_2(X) \\ \varphi_3(X) \\ \vdots \\ \varphi_r(X) \end{pmatrix} = \begin{pmatrix} L_F^0 h \\ L_F^1 h \\ L_F^2 h \\ \vdots \\ L_F^{r-1} h \end{pmatrix}. \tag{42}$$

Define the new input ν which linearises the the system as follows:

$$\sigma = L_F^r h, \tag{43}$$
$$\omega = L_G(L_F^{r-1} h), \tag{44}$$
$$u = \frac{1}{\omega(X)}[-\sigma(X) + \nu]. \tag{45}$$

2.4.3 Feedback Linearization for EMBS System Model with the input voltage

Theorem 2.7. *The EMBS model with the voltage as input, given by the system* (6) *is input-state linearisable.*

Proof. To prove this it is sufficient to show that the vector field $\{G, ad_F G, ..., ad_F^{n-1} G\}$ is linearly independent in D, and the vector field $\{G, ad_F G, ..., ad_F^{n-2} G\}$ is involutive in D, where G and F are given by the following equations:

$$G = \begin{pmatrix} \frac{1}{L} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \tag{46}$$

$$F = \begin{pmatrix} -\frac{Rx_1}{L} \\ x_3 \\ -g + \frac{kx_2}{m} - \frac{\beta x_3}{m} + \frac{\alpha x_1^2}{m x_2^2} \end{pmatrix}. \tag{47}$$

The observation that should be made here, the $n = 3$.

$$ad_F G = \begin{pmatrix} \frac{R}{L^2} \\ 0 \\ -\frac{2\,\alpha x_1}{L m x_2^2} \end{pmatrix}, \tag{48}$$

$$ad_F^2 G = \begin{pmatrix} \frac{R^2}{L^3} \\ \frac{2\,\alpha x_1}{L m x_2^2} \\ -\frac{2\,((\alpha\beta x_1 x_2 - 2\,\alpha m x_1 x_3)L)}{L^2 m^2 x_2^3} \end{pmatrix}. \tag{49}$$

We can check that the vector field

$$\{G, ad_F G, ad_F^2 G\}$$

is linearly independent by finding the determinant of the following matrix:

$$\begin{pmatrix} \frac{1}{L} & 0 & 0 \\ \frac{R}{L^2} & 0 & -\frac{2\,\alpha x_1}{L m x_2^2} \\ \frac{R^2}{L^3} & \frac{2\,\alpha x_1}{L m x_2^2} & -\frac{2\,(V\alpha m x_2 + (\alpha\beta x_1 x_2 - 2\,\alpha m x_1 x_3)L)}{L^2 m^2 x_2^3} \end{pmatrix}.$$

The determinant of this matrix is given by:

$$\frac{4\,\alpha^2 x_1^2}{L^3 m^2 x_2^4}. \tag{50}$$

If variables x_1 and x_2 in the domain D are different from zero then

$$\{G, ad_F G, ad_F^2 G, ad_F^3 G\}$$

is linearly independent. It remains to show that the vector field

$$\{G, ad_F G\}$$

is involute. One shows that the following vectors

$$[G, G], [G, ad_F G]$$

are linear combination of vectors

$$\{G, ad_F G\}.$$

This is true because other vectors are zero or opposite of these vectors above. Now

$$[G, G] = (0, 0, 0)^T,$$

and

$$[ad_F G, ad_F G] = (0, 0, \frac{-2\alpha}{L^2 m x_2^2})^T.$$

It convenient to show only that $[G, ad_F G]$ is a linear combination of vectors

$$\{G, ad_F G\}$$

. One can write $G, ad_F G]$ as follows:

$$[G, ad_F G] = \frac{-R}{L^2 x_1} G - \frac{1}{L x_1} ad_F G. \tag{51}$$

Thus the vector fields

$$\{G, ad_F G\}$$

is involutive, since the relative degree of this model is 3 when $x_1 \neq 0$ and $x_2 \neq 0$. Hence the system is input-output state linearizable.

2.4.4 Feedback Linearized Model for EMBS System with Voltage as Input After realizing that the EMBSS with voltage input is feedback linearizable with relative three, one proceeds with the linearization stage. The information given by (42), (43), (44) and (45) are going to be used here as follows:

$$z_1 = L_F^0 h = x_2, \tag{52}$$
$$z_2 = L_F^1 h = x_3, \tag{53}$$
$$z_3 = L_F^2 h,$$
$$= -g + \frac{\alpha}{m}(\frac{x_1}{x_2})^2 + \frac{k}{m}x_2 - \frac{\beta}{m}x_3, \tag{54}$$
$$\sigma = -\frac{2\,R\alpha m x_1^2 x_2}{L m^2 x_2^3} - \frac{\left(\beta g m x_2^2 - \beta k x_2^3 - \alpha\beta x_1^2\right)}{m^2 x_2^2} - \frac{\left(2\alpha m x_1^2 - \left(\beta^2 + km\right) x_2^3\right) x_3 L}{L m^2 x_2^3}, \tag{55}$$
$$\omega = \frac{2\alpha x_1}{L m x_2^2}, \tag{56}$$
$$z_4 = \sigma(X) + \omega(X)u. \tag{57}$$

It is clear to observe the following:

$$\frac{dz_1}{dt} = z_2,$$
$$\frac{dz_2}{dt} = z_3,$$
$$\frac{dz_3}{dt} = z_4 = \sigma(X) + \omega(X)u. \tag{58}$$

Putting equation (45) in equation (58), gives the following:

$$\frac{dz_1}{dt} = z_2, \tag{59}$$
$$\frac{dz_2}{dt} = z_3, \tag{60}$$
$$\frac{dz_3}{dt} = \nu. \tag{61}$$

In matrix form this can be written as follows:

$$\begin{pmatrix} \frac{dz_1}{dt} \\ \frac{dz_2}{dt} \\ \frac{dz_3}{dt} \end{pmatrix} = \begin{pmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 0\,0\,0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \nu. \tag{62}$$

Remark:
Here the input of the system is ν, and the output of the system is $Z_1 = x_2$. The input ν can be chosen in such away that the solution of the system is asymptotically stable.

2.4.5 Feedback Linearization for EMBS System Model with the Input Current

Theorem 2.8. *The EMBS model with the current as input, given by the system* (12) *is output-input state linearisable on the domain D where* $x_1 \neq 0$ *and* $x_3 \neq 0$.

Proof. To prove this it is sufficient to show that the vector field $\{G, ad_F G, ..., ad_F^{n-1} G\}$ is linearly independent in D, and the vector field $\{G, ad_F G, ..., ad_F^{n-2} G\}$ is involutive in D. Where G and F are given by the following equations:

$$G = \begin{pmatrix} 0 \\ \frac{1}{CL} \\ 0 \\ 0 \end{pmatrix} \tag{63}$$

$$F = \begin{pmatrix} x_2 \\ -\frac{R_2 x_2}{L} + \frac{I}{CL} - \frac{x_1}{CL} \\ x_4 \\ -g + \frac{k x_3}{m} - \frac{\beta x_4}{m} + \frac{\alpha x_1^2}{m x_3^2} \end{pmatrix}. \tag{64}$$

The observation that should be made here, the $n = 4$.

$$ad_F G = \begin{pmatrix} -\frac{1}{CL} \\ \frac{R_2}{CL^2} \\ 0 \\ 0 \end{pmatrix}, \tag{65}$$

$$ad_F^2 G = \begin{pmatrix} -\frac{R_2}{CL^2} \\ \frac{R_2^2}{CL^3} - \frac{1}{C^2L^2} \\ 0 \\ \frac{2\,\alpha x_1}{CLm x_3^2} \end{pmatrix} \quad \text{and} \tag{66}$$

$$ad_F^3 G = \begin{pmatrix} -\frac{CR_2^2 - L}{C^2L^3} \\ \frac{CR_2^3 - 2\,LR_2}{C^2L^4} \\ -\frac{2\,\alpha x_1}{CLm x_3^2} \\ \pi \end{pmatrix} \tag{67}$$

Here

$$\pi = \frac{2\left(R_2 \alpha m x_1 x_3\right)}{CL^2 m^2 x_3^3} - \frac{\left(\left(2\alpha m x_1 x_4 - \left(\alpha\beta x_1 + \alpha m x_2\right) x_3 L\right)\right)}{CL^2 m^2 x_3^3}$$

We can check that the vector field

$$\{G, ad_F G, ad_F^2 G, ad_F^3 G\}$$

is linearly independent by finding the determinant of the following matrix:

$$\begin{pmatrix} 0 & \frac{1}{CL} & 0 & 0 \\ -\frac{1}{CL} & \frac{R_2}{CL^2} & 0 & 0 \\ -\frac{R_2}{CL^2} & \frac{R_2^2}{CL^3} - \frac{1}{C^2L^2} & 0 & \frac{2\,\alpha x_1}{CLm x_3^2} \\ -\frac{R_2^2}{CL^3} + \frac{1}{C^2L^2} & \frac{R_2\left(\frac{R_2^2}{CL^3} - \frac{1}{C^2L^2}\right)}{L} - \frac{R_2}{C^2L^3} & -\frac{2\,\alpha x_1}{CLm x_3^2} & \Psi \end{pmatrix},$$

where

$$\Psi = \frac{2\,\alpha\beta x_1}{CLm^2 x_3^2} + \frac{2\,R_2 \alpha x_1}{CL^2 m x_3^2} + \frac{2\,\alpha x_2}{CLm x_3^2} - \frac{4\,\alpha x_1 x_4}{CLm x_3^3}.$$

The determinant of this matrix is given by:

$$\frac{4\,\alpha^2 x_1^2}{C^4 L^4 m^2 x_3^4}. \tag{68}$$

If variables x_1 and x_3 in the domain D are different from zero then

$$\{G, ad_F G, ad_F^2 G, ad_F^3 G\}$$

is linearly independent. It remains to show that the vector field

$$\{G, ad_F G, ad_F^2 G\}$$

is involute. One shows that the following vectors

$$[G,G], [G, ad_F G], [G, ad_F^2 G], [ad_F G, ad_F^2 G]$$

are linear combination of vectors

$$\{G, ad_F G, ad_F^2 G\}.$$

This is true because other vectors are zero or opposite of these vectors above. Now

$$[G,G] = (0,0,0,0)^T,$$

$$[G, ad_F G] = (0,0,0,0)^T,$$

and

$$[ad_F G, ad_F^2 G] = (0,0,0,\frac{-2\alpha}{C^2 L^2 m x_3^2})^T.$$

It convenient to show only that $[ad_F G, ad_F^2 G]$ is a linear combination of vectors

$$\{G, ad_F G, ad_F^2 G\}.$$

One can write $[ad_F G, ad_F^2 G]$ as follows:

$$\begin{aligned}[ad_F G, ad_F^2 G] = & \frac{1}{C^2 L^2 x_1} G - \frac{R_2}{L^2 C x_1} ad_F G \\ & + \frac{1}{L C x_1} ad_F^2 G. \end{aligned} \tag{69}$$

Thus the vector fields

$$\{G, ad_F G, ad_F^2 G\}$$

is involutive, since the relative degree of this model is 4 when $x_1 \neq 0$ and $x_3 \neq 0$. Hence the system is input-output state linearizable.

2.4.6 Feedback Linearized Model for EMBS System with the Current Input After realizing that the EMBSS with current input is feedback linearizable with relative four, one proceeds with the linearization stage. The information given by (42), (43), (44) and (45) are going to be used here as follows:

$$\begin{aligned}
z_1 &= L_F^0 h = x_3, \\
z_2 &= L_F^1 h = x_4, \\
z_3 &= L_F^2 h = -g + \frac{\alpha}{m}(\frac{x_1}{x_2})^2 + \frac{k}{m} x_2 - \frac{\beta}{m} x_3, \\
z_4 &= L_F^3 h, \\
&= \frac{\beta g m x_3^3 - \beta k x_3^4 - \big(\alpha\beta x_1^2 - 2\,\alpha m x_1 x_2\big) x_3}{m^2 x_3^3} \\
&\quad + \frac{-\big(2\,\alpha m x_1^2 - \big(\beta^2 + km\big) x_3^3\big) x_4}{m^2 x_3^3}, \\
\sigma &= L_F^4 h, \\
&= -\frac{2\,C R_2 \alpha m^2 x_1 x_2 x_3^3 - 2\,I \alpha m^2 x_1 x_3^3 + 2\,\alpha m^2 x_1^2 x_3^3}{C L m^3 x_3^5} \\
&\quad - \frac{\big(2\,\alpha g m^2 x_1^2 x_3^2 + 6\,\alpha m^2 x_1^2 x_3 x_4^2 - 2\,\alpha^2 m x_1^4\big)}{m^3 x_3^5} \\
&\quad + \frac{\big(\beta^2 k + k^2 m\big)\, x_3^6 - \big(\beta^2 g m + g k m^2\big)\, x_3^5}{m^3 x_3^5} \\
&\quad + \frac{-\,\big(2\,\alpha\beta m x_1 x_2 - 2\,\alpha m^2 x_2^2\big)}{m^3 x_3^5} \\
&\quad - \frac{\big(\big(\alpha\beta^2 - \alpha k m\big) x_1^2\big) x_3^3 - \big(\big(\beta^3 + 2\,\beta k m\big) x_3^5 x_4\big)}{m^3 x_3^5} \\
&\quad + \frac{-4\big(\alpha\beta m x_1^2 - 2\,\alpha m^2 x_1 x_2\big) x_4}{m^3 x_3^3}, && (70) \\
\omega &= -\frac{2\,(R_2 \alpha m x_1 x_3 + (4\,\alpha m x_1 x_4 + (\alpha\beta x_1 - 2\,\alpha m x_2) x_3) L)}{C L^2 m^2 x_3^3}, && (71) \\
z_5 &= \sigma(X) + \omega(X) u. && (72)
\end{aligned}$$

It is clear to observe the following:

$$\begin{aligned}
\frac{dz_1}{dt} &= z_2, \\
\frac{dz_2}{dt} &= z_3, \\
\frac{dz_3}{dt} &= z_4, \\
\frac{dz_4}{dt} &= z_5 = \sigma(X) + \omega(X) u. && (73)
\end{aligned}$$

Putting equation (45) in equation (76), gives the following:

$$\begin{aligned}
\frac{dz_1}{dt} &= z_2, && (74) \\
\frac{dz_2}{dt} &= z_3, && (75) \\
\frac{dz_3}{dt} &= z_4, \\
\frac{dz_3}{dt} &= \nu. && (76)
\end{aligned}$$

In matrix form this can be written as follows:

$$\begin{pmatrix} \frac{dz_1}{dt} \\ \frac{dz_2}{dt} \\ \frac{dz_3}{dt} \\ \frac{dz_4}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \nu. \quad (77)$$

Remark: Here the input of the system is ν, and the output of the system is $Z_1 = x_3$. The input ν can be chosen in such away that the solution of the system is asymptotically stable.

2.5 Optimal Control

The choice of the input of the control system is not always is, this applies also to our formulated models. One need a technique which will help us to find the optimal control law. One shall perform the optimal control technique known as linear quadratic regulator (LQR) [Basile and Marro, 1992]. One will need to minimise the objective function

$$J(U) = \int_{t_0}^{\infty} (X^T Q X + U^T R U) dt, \quad (78)$$

subjected to the equation of the form (62) or (77), where Q and R are positive definite matrices. That is,

$$\dot{X} = AX + BU, \quad (79)$$

where A and B are time invariant matrices and U is the input. The required solution is the input U which minimises equation (78). The optimal control input is given by $U = KX$ where $K = -R^{-1}B^T P$ and P is obtained by solving an algebraic Riccati equation given as follows: $A^T P + PA - PBR^{-1}B^T P + Q = 0$. The following table will be used for parameter values in the two models:

Parameters	R_2	L	m	α	β	k
Values	1	1	0.5	0.0001	0.8	3
Units	Ω	Henry	kg	$\frac{Nm^2}{A^2}$	$\frac{Ns}{m}$	$\frac{N}{m}$

Parameters	C	x_{3e}	g	R	x_2
Values	0.5	3	9.81	1	0.5
Units	Farad	m	$\frac{m}{s^2}$	Ω	m

2.5.1 Optimal Control for Feedback Linearized EMBSS Model with Voltage Input

The feedback linearized model given by the equation (62) is being optimized in this section. One needs to chose positive matrices R and Q. Let them be $R = 100$ and

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (80)$$

The remaining is to solve for P in the Riccati's equation $A^T P + PA - PBR^{-1}B^T P + Q = 0$. The matrices A and B are given by the equations (81) and (82) as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (81)$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (82)$$

The octave function $lqr()$ is used to produce the following matrix:

$$P = \begin{pmatrix} 6.4338 & 20.1967 & 31.6228 \\ 20.1967 & 98.3184 & 203.4538 \\ 31.6228 & 203.4538 & 638.6764 \end{pmatrix}. \quad (83)$$

Then K is calculated using the formula, $K = -R^{-1}B^T P$ to obtain the following:

$$K = \begin{pmatrix} 0.031623 & 0.203454 & 0.638676 \end{pmatrix}. \quad (84)$$

The input of the system is then calculated by KX, where

$$X = \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix}^T.$$

The following figure is the simulation of the optimized feedback Linearized model of EMBSS when the input is the voltage.

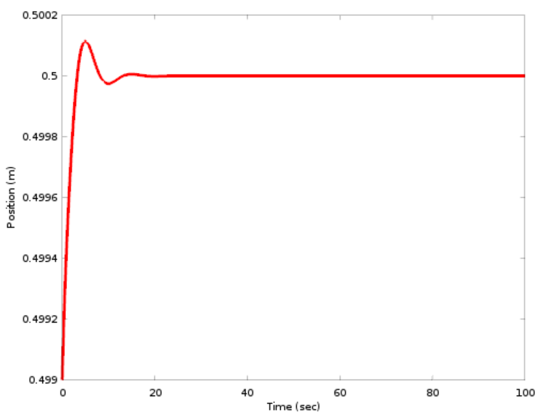

Figure 3. : Optimal control for feedback linearized model of EMBSS with current input.

2.5.2 Optimal Control for Feedback Linearized EMBSS Model with Current Input

The feedback linearized model given by the equation (77) is being optimized in this section. One needs to chose positive matrices R and Q. Let them be $R = 100$ and

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (85)$$

The remaining is to solve for P in the Riccati's equation $A^T P + PA - PBR^{-1}B^T P + Q = 0$. The matrices A and B are given by the equations (86) and (87) as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \qquad (86)$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T. \qquad (87)$$

The octave function $lqr()$ is used to produce the following matrix:

$$P = \begin{pmatrix} 6.0790 & 17.9773 & 28.5797 & 31.6228 \\ 17.9773 & 80.7047 & 161.0914 & 192.2357 \\ 28.5797 & 161.0914 & 402.2549 & 568.4918 \\ 31.6228 & 192.2357 & 568.4918 & 1096.0060 \end{pmatrix}.$$

Then K is calculated using the formula, $K = -R^{-1}B^T P$ to obtain the following:

$$K = \begin{pmatrix} 0.031623 & 0.192236 & 0.568492 & 1.096006 \end{pmatrix}.$$

The input of the system is then calculated by KX, where $X = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \end{pmatrix}^T$. The following figure is the simulation of the optimized feedback Linearized model of EMBSS when the input is the current.

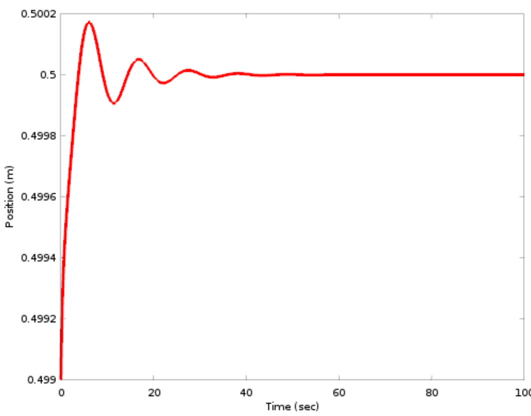

Figure 4. : Optimal control for feedback linearized model of EMBSS with current input.

3 Conclusion

Two models were built depending on whether the input is voltage or current. The two models were represented in state representation form. The feedback linearization techniques were applied to completely linearize the two non linear models. The optimal control input for both models was given by solution of algebraic Riccati equation. It was found out that the two models formulated are input-state feedback linearizable.

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