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ON ANALYTICAL EVALUATIONS OF TRIPLE INTEGRALS IN COMPLEX DOMAINS

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On Analytical Evaluations of Triple Integrals in Complex Domains

Synopsis:

Evaluation of triple integrals in complex volumes could be difficult for college students due to several complexities. This paper presents several alternatives for setting up triple integrals in complex volumes and perspectives for assessing triple integrals with diverse coordinate. Not only could these provide cross-checks for complex integrals, but also stimulate students' enthusiasm to sparkle thoughts after class so as to enhance the understanding of triple integral evaluations.

On Analytical Evaluations of Triple Integrals in Complex Domains

Abstract

Evaluation of triple integrals in complex volumes could be difficult for college students due to the complexities in the integrands and the analytical expressions of the multi-component domains. This paper outlines overall strategies and presents several alternatives for setting up triple integrals in complex volumes and perspectives for assessing triple integrals with diverse coordinate systems. Not only could these provide cross-checks for evaluating triple integrals within complex domains, but also stimulate enthusiasm and curiosity of students to sparkle follow-up thoughts so as to enhance the understanding of evaluating triple integral in complex domains.

Key words: Triple integrals; Cartesian coordinates; Cylindrical coordinaes; Spherical coordinates

1 Overview

The triple integrals discussed here are Riemann integrals, as defined in [1,2], which are to be evaluated analytically as opposed to approximately in numerical classes.

Evaluation of triple integrals in complex volumes is one of the most difficult sections in calculus series and is challenging for most college students. Students usually have trouble setting up integrals and face problems about which coordinate system to choose and what values to set for limits of integral variables. In addition, students could also encounter problems about which variable goes to the inner integral and which option would be easier to evaluate.

Usually, a triple integral evaluation should be set up such that the inner integral is from one surface to another surface, the middle integral is from one curve to another curve, and the outer integral is from one constant to another. Occasionally, the inner surface integral could be from one tilted plane to another and the middle integral could be from a tilted straight line to another. But the outer integral usually is a definite integral unless otherwise. If cylindrical polar coordinate system is used, usually the vertical direction z is integrated first before setting up a polar integral. Some students could also make mistakes in matching integrals limits with the corresponding variables, especially during the projection of the domain into two other variables after the inner variable is decided.

Sometimes integrands or domains are complicated that the conventional coordinate systems may seem clumsy to handle integrals. A change of variables with a Jacobian could make a difference. Although this could simplify the integrand, this could also lead to irregularly complex domains for one or more integrals. On the other hand, changing variables could achieve a regularized domain but at the price of a cumbersome integrand which jeopardizes the objective [3].

The next two sections provide several alternatives for setting up triple integrals in complex volumes and different ideas of assessing triple integrals with diverse coordinate systems. The objective here is to provide cross-checks for evaluating triple integrals over complex domains while stimulate enthusiasm and curiosity of students to sparkle further thoughts after class meetings, to assist them using better approaches to evaluate triple integral on complex domains.

2 Inner Variable and Area of Projection

Cartesian coordinate system is commonly used in many problems. Cases involving cylindrical and spherical coordinate systems are to be discussed in a later Section. The focus here is dedicated towards Cartesian coordinate system. Regarding questions of which direction should be integrated first and which variable shall be the inner variable, two factors are usually considered: the shape of the domain and the form of the integrand. In order to simplify the setup and the evaluation process, usually it is beneficial to select the variable which allows the domain starts from one entire surface to another and to project the domain into the plane of the other two variable, in an effort to simplify analytical expressions as much as possible. This could avoid two things: Multiple integrals setups and difficult evaluations. The following example problem demonstrates the differences.

Evaluate the triple integral, which is similar to the one in [4]:

$$\iiint_{\Omega} z \, d\Omega,$$

44 where Ω is bounded by the circular cylinder $y^2 + z^2 = 256$, the plane $x = 0$, the plane $y = 4x$, and the plane $z = 0$, as shown in Figure 1:

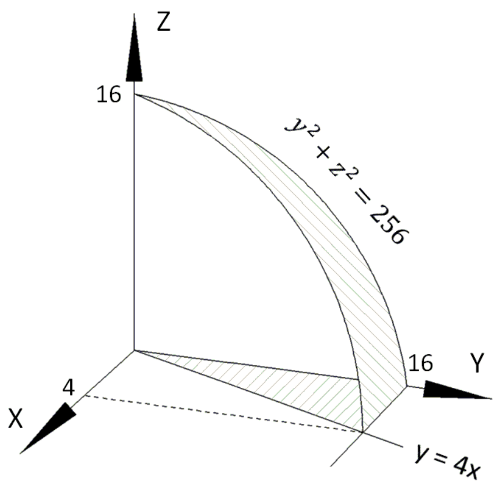

Fig. 1. Sketch of the top part of the domain, i.e., part of the circular cylinder $y^2 + z^2 = 256$, and the bottom part of the domain, i.e., triangle in shadow, and the volume surrounded by the circular cylinder and three planes, $x = 0$, $y = 4x$, and $z = 0$

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46 *2.1 Integrate x First: The Best Choice*

Choosing x as the inner variable and integrating x before the other two is preferred. The volume starts from the **surface** $x = 0$ within the circle in y-z plane and ends at the tilted **surface** $x = y/4$, as shown in Figure 1. Next, the volume is projected into the y-z plane, and the projected area is a solid circle centered at the origin within the first quadrant of the principal y-z plane. It does not make much difference as to y or z to be the middle variable. Then, the middle integral sweeps through the **curve** $y = 0$ to the **curve** $y = \sqrt{256 - z^2}$. Finally, the outer integral is from the **constant**

$z = 0$ to the **constant** $z = 16$. The triple integral is evaluated as:

$$
\begin{aligned}
V &= \iiint_\Omega z \, d\Omega, \\
&= \int_0^{16} \int_0^{\sqrt{256-z^2}} \int_0^{\frac{y}{4}} z \, dx dy dz \\
&= \int_0^{16} \int_0^{\sqrt{256-z^2}} z \, \frac{y}{4} \, dy dz \\
&= \int_0^{16} \frac{z}{8} \, y^2 \Big|_0^{\sqrt{256-z^2}} dz \\
&= \int_0^{16} \left(32z - \frac{z^3}{8} \right) dz \\
&= \left(16z^2 - \frac{z^4}{32} \right)_0^{16} = 4096 - 2048 \\
&= 2048.
\end{aligned} \tag{1}
$$

An alternative here is to switch to polar coordinate system once x is integrated. The process is shown below:

$$
\begin{aligned}
V &= \iiint_\Omega z \, d\Omega, \\
&= \int_0^{16} \int_0^{\sqrt{256-z^2}} \int_0^{\frac{y}{4}} z \, dx dy dz \\
&= \int_0^{16} \int_0^{\sqrt{256-z^2}} z \, \frac{y}{4} \, dy dz \\
&= \frac{1}{4} \int_0^{\frac{\pi}{2}} \int_0^{16} r \sin(\theta) r \cos(\theta) r dr d\theta \\
&= \frac{1}{4} \int_0^{\frac{\pi}{2}} \sin(\theta) \cos(\theta) d\theta \int_0^{16} r^3 dr \\
&= 2048.
\end{aligned} \tag{2}
$$

This could lead to exactly the same answer and less effort.

2.2 Integrate z First: The Better Choice

If z is chosen as the inner variable, the volume starts from the shaded triangular **surface** $z = 0$ and ends at the **surface** $y^2 + z^2 = 256$, as shown in Figure 1. Next, the volume is projected into the x-y plane, and the projected area is the shaded solid triangle in this figure. It would be essentially the same for x or y to be the middle variable. Then, the middle integral for y sweeps through the tilted **line** $y = 4x$ to the **line** $y = 16$. Finally, the outer integral is from the **constant** $x = 0$ to the **constant**

$x = 4$. The triple integral can be assessed as below:

$$
\begin{aligned}
V &= \iiint_{\Omega} z \, d\Omega, \\
&= \int_0^4 \int_{4x}^{16} \int_0^{\sqrt{256-y^2}} z \, dz dy dx \\
&= \int_0^4 \int_{4x}^{16} \left(128 - \frac{y^2}{2}\right) dy dx \\
&= \int_0^4 \left(128y - \frac{y^3}{6}\right)_{4x}^{16} dx \\
&= \int_0^4 \left(\frac{4096}{3} - 512x + \frac{32}{3}x^3\right) dx \\
&= \frac{16384}{3} - 4096 + \frac{2048}{3} = 6144 - 4096 \\
&= 2048.
\end{aligned}
\tag{3}
$$

49 *2.3 Integrate y First: The Least Desired Choice*

Finally, if y is chosen as the inner variable, then the volume starts from the tilted **plane** $y = 4x$ inscribed by the circular cylinder $y^2+z^2 = 256$, and ends at this circular **cylinder** $y^2 + z^2 = 256$ in the first octant truncated by the tilted plane $y = 4x$, as shown in Figure 1. Next, the volume is projected into the x-z plane, and the projected area can be determined by solving below two equations together:

$$
\begin{cases} y^2 + z^2 &= 256, \\ y &= 4x, \end{cases} \qquad \text{which gives } \; 16\,x^2 + z^2 = 256. \tag{4}
$$

Therefore, the projected area onto the x-z plane is one quarter of the solid ellipse enclosed by the lines $x = 0$, $z = 0$, and the ellipse $16x^2 + z^2 = 256$. Then, either x or z can be the middle variable. If z is integrated next, the middle z integral sweeps through the **line** $z = 0$ to the **ellipse** $z = \sqrt{256 - 16x^2}$. Finally, the outer integral is for x from the **constant** $x = 0$ to the **constant** $x = 4$. The triple integral can be assessed as below:

$$
\begin{aligned}
V &= \iiint_{\Omega} z\, d\Omega, \\
&= \int_0^4 \int_0^{\sqrt{256-16x^2}} \int_{4x}^{\sqrt{256-z^2}} z\, dydzdx \\
&= \int_0^4 \int_0^{\sqrt{256-16x^2}} \left(z\sqrt{256-16z^2} - 4xz \right) dzdx \\
&= \int_0^4 \left(\frac{1}{3}\left(256-z^2\right)^{\frac{3}{2}} \Big|_{\sqrt{256-16x^2}}^{0} - 2xz^2 \Big|_0^{\sqrt{256-16x^2}} \right) dx \\
&= \int_0^4 \left(\frac{4096}{3} - \frac{64}{3}x^3 - 2x\left(256-16x^2\right) \right) dx \\
&= \left(\frac{4096}{3}x - 256x^2 + \frac{8}{3}x^4 \right)_0^4 = \frac{16384+2048}{3} - 4096 = 6114 - 4096 \\
&= 2048.
\end{aligned}
\tag{5}
$$

Although the same outcome follows, it is the most involved approach.

3 Right Coordinate System and Suitable Integral Form

In triple integrals, if expressions such as $x^2 + y^2$, $y^2 + z^2$, or $x^2 + z^2$ appear in the description of domain and the integrand, usually it is beneficial to use cylindrical polar coordinate system. But if $x^2 + y^2 + z^2$ appears either in the domain or the integrand, then spherical coordinate system could be much better. The choice of coordinate system (CS) is of importance to the evaluation of triple integrals. Although no CS works for all problems, usually there is a better CS to use for a specific problem. Therefore, the choice of CS really depends on the shape of the volume where triple integrals are defined and the form of the integrand that is to be integrated. The following problem demonstrates that different CS could be used for the same problem. However, the setup and the effort to evaluate triple integrals could vary significantly.

Example problem: Find the volume within the sphere $x^2 + y^2 + z^2 = 49$, above the plane $z = 0$ and outside the cone $z = 3\sqrt{x^2 + y^2}$. It would be very difficult to evaluate the volume with Cartesian CS because of the curvature of the solid involved.

3.1 Difference of Two Cylindrical Triple Integrals

By cylindrical polar coordinates [5], the equations for the upper cone and the upper sphere are $z = 3r$ and $r^2 + z^2 = 49$, respectively. The volume to be found equals the volume of the upper half sphere without the volume V_0, which is the volume above the upper half cone and within the upper half sphere, as shown in Figure 2.

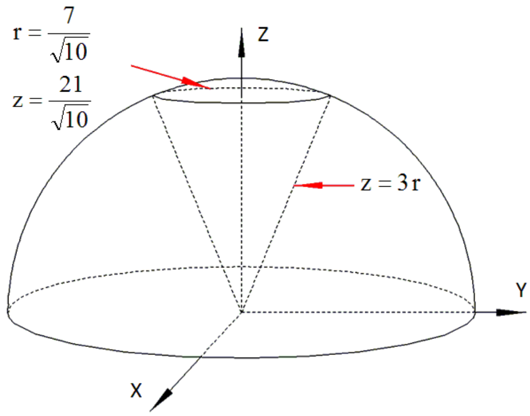

Fig. 2. Sketch of the geometry of the upper half cone, the upper half sphere, and the intersection given in cylindrical polar coordinates

First, the volume of V_0 is assessed. By squaring the equation of the cone and substituting $z^2 = 9r^2$ into the equation of the sphere $r^2 + z^2 = 49$, the intersectional curve between them is found to be $r = 7/\sqrt{10}$ and $z = 21/\sqrt{10}$. Therefore, limits for the triple integral about the volume of V_0 are:

$$\begin{cases} 3r \le z \le \sqrt{49 - r^2}, \\ 0 \le r \le \frac{7}{\sqrt{10}}, \\ 0 \le \theta \le 2\pi. \end{cases} \tag{6}$$

Integrate z first, which is from the cone to the sphere, the volume of V_0 is then evaluated as below:

$$\begin{aligned} V_0 &= \int_0^{2\pi} \int_0^{\frac{7}{\sqrt{10}}} \int_{3r}^{\sqrt{49-r^2}} dz\, r dr\, d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{7}{\sqrt{10}}} (\sqrt{49 - r^2} - 3r)\, r dr\, d\theta \\ &= \frac{2\pi}{3}(49 - r^2)^{3/2}\Big|_{\frac{7}{\sqrt{10}}}^{0} - 2\pi \frac{7^3}{10\sqrt{10}} \\ &= \frac{2\pi}{3} 7^3 - \frac{2\pi}{\sqrt{10}} 7^3. \end{aligned} \tag{7}$$

The volume V equals the volume of the upper half sphere subtracting that of V_0:

$$V = \frac{1}{2}\left(\frac{4\pi}{3} 7^3\right) - \left(\frac{2\pi}{3} 7^3 - \frac{2\pi}{\sqrt{10}} 7^3\right) = \frac{2\pi}{\sqrt{10}} 7^3. \tag{8}$$

3.2 Summation of Two Cylindrical Triple Integrals

Using cylindrical coordinates as well, with previous knowledge about the intersectional curve, the upper cone $z = 3r$ and the upper half sphere is $r^2 + z^2 = 49$, the desired volume could be found by adding two volumes together. The first one, denoted as V_1, is inside the circular cylinder $r = 7/\sqrt{10}$ and outside the upper cone $z = 3r$. The second volume, denoted as V_2, is outside the circular cylinder $r = 7/\sqrt{10}$ and inside the upper half sphere $r^2 + z^2 = 49$. See Figure 3 for detail.

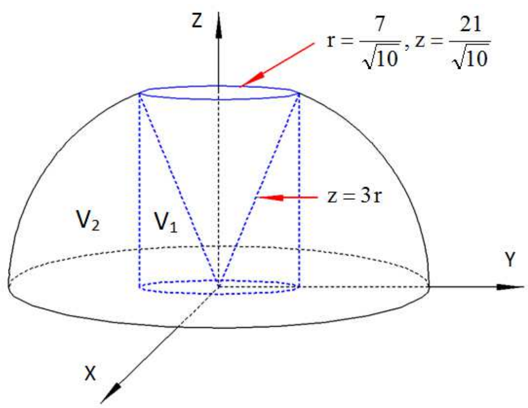

Fig. 3. Sketch of the circular cylinder, the upper half cone, and the upper sphere in cylindrical coordinates: the desired volume $V = V_1 + V_2$

Integrate z first in the following cylindrical integral, V_1 can be evaluated as:

$$V_1 = \int_0^{2\pi} \int_0^{\frac{7}{\sqrt{10}}} \int_0^{3r} dz\, r dr\, d\theta = \int_0^{2\pi} \int_0^{\frac{7}{\sqrt{10}}} 3r^2 dr\, d\theta = \frac{2\pi}{10\sqrt{10}} 7^3. \tag{9}$$

Integrate z first as well and assess the integral for V_2 accordingly:

$$V_2 = \int_0^{2\pi} \int_{\frac{7}{\sqrt{10}}}^{7} \int_0^{\sqrt{49-r^2}} dz\, r dr\, d\theta = \int_0^{2\pi} \int_{\frac{7}{\sqrt{10}}}^{7} \sqrt{49 - r^2}\, r dr\, d\theta = \frac{9(2\pi)}{10\sqrt{10}} 7^3. \tag{10}$$

Finally, the desired volume is the sum of V_1 and V_2:

$$V = V_1 + V_2 = \frac{2\pi}{10\sqrt{10}} 7^3 + \frac{9(2\pi)}{10\sqrt{10}} 7^3 = \frac{2\pi}{\sqrt{10}} 7^3. \tag{11}$$

3.3 Spherical Coordinates Triple Integral

Using spherical coordinates [5], the radial distance is ρ, the polar angle is ϕ, and the azimuthal angle is θ. The representation of the upper half sphere becomes $\rho = 7$ and $0 \le \phi \le \pi/2$. The equation of the cone is $\rho \cos(\phi) = 3\rho \sin(\phi)$.

In order to set up the triple integral for the volume, it is necessary to ascertain the equation for the intersection of the upper half sphere and the cone. To this end, the equation of the cone is squared such that $\cos^2(\phi) = 9\sin^2(\phi)$. Because of the trigonometric identity, $sin^2(\phi)+cos^2(\phi)=1$, the equation of the intersection becomes:

$$\rho = 7 \quad \text{and} \quad \cos(\phi) = \frac{3}{\sqrt{10}}, \quad \text{i.e.,} \quad \phi = \cos^{-1}\frac{3}{\sqrt{10}}. \tag{12}$$

Limits of the triple integral over the desired volume are summarized as:

$$\begin{cases} 0 \le \rho \le 7, \\ \cos^{-1}\frac{3}{\sqrt{10}} \le \phi \le \frac{\pi}{2}, \\ 0 \le \theta \le 2\pi. \end{cases} \tag{13}$$

Because all limits are constants, as long as integral variables and associated limits match well, the volume V could be found via below triple integral:

$$\begin{aligned} V &= \int_0^7 \int_0^{2\pi} \int_{\cos^{-1}\frac{3}{\sqrt{10}}}^{\frac{\pi}{2}} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho \\ &= 2\pi \int_0^7 \rho^2 d\rho \int_{\cos^{-1}\frac{3}{\sqrt{10}}}^{\frac{\pi}{2}} \sin\phi \, d\phi \\ &= 2\pi \frac{7^3}{3} \cos\phi \Big|_{\frac{\pi}{2}}^{\cos^{-1}\frac{3}{\sqrt{10}}} \\ &= \frac{2\pi}{\sqrt{10}} 7^3. \end{aligned} \tag{14}$$

Apparently, using spherical coordinates, the integral only involves constant limits, which is easier than previous two ways.

3.4 One Line Integrals of Piled-up Rings

The volume of the solid region could be evaluated exactly by adding up a series of variable-sized rings of thickness dz in a **line integral**, i.e., a single variable integral, under cylindrical coordinates. The inner radius of the ring, R_{in} is specified by the cone $z = 3r$ as $R_{in} = z/3$. The outer radius of the ring, R_{out} is bounded by the upper half sphere $r^2 + z^2 = 49$; therefore, $R_{out} = \sqrt{49 - z^2}$.

These rings pile up from the bottom of the upper half sphere, $z = 0$, to the heigh of

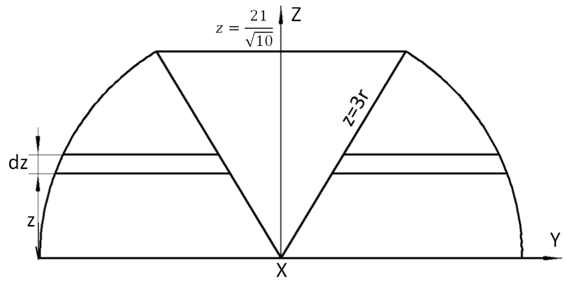

Fig. 4. Line integral evaluation: Side view of the upper half cone and the upper sphere in a cylindrical coordinate system

the intersection $z = 21/\sqrt{10}$. The volume is evaluated with one line integral below:

$$\begin{aligned}
V &= \int_0^{\frac{21}{\sqrt{10}}} \pi(R_{out}^2 - R_{in}^2)\, dz \\
&= \pi \int_0^{\frac{21}{\sqrt{10}}} \left[(49 - z^2) - \frac{z^2}{9}\right] dz \\
&= \pi \int_0^{\frac{21}{\sqrt{10}}} \left[49 - \frac{10z^2}{9}\right] dz \\
&= 49\pi z\Big|_0^{\frac{21}{\sqrt{10}}} - \frac{10\pi}{27} z^3\Big|_0^{\frac{21}{\sqrt{10}}} \\
&= \frac{3\pi}{\sqrt{10}}\, 7^3 - \frac{\pi}{\sqrt{10}}\, 7^3 \\
&= \frac{2\pi}{\sqrt{10}}\, 7^3.
\end{aligned} \tag{15}$$

This seems to be even easier than using the spherical coordinates.

3.5 Summation of Line Integrals of Disks

Using the similar idea as in Subsection 3.1, the same volume to be determined equals to the volume of upper half sphere subtracting the volume of V_0 as defined in Subsection 3.1. However, several simple line integrals are used here to find V_0, and then evaluate the desired volume V with cylindrical coordinates, see Figure 5 for details.

To facilitate the evaluation of these integrals, the volume V_0 is divided into two parts: the flat cone V_1 under the plane $z = 21/\sqrt{10}$ and within the cone $z = 3r$; and the crown V_2 above the plane $z = 21/\sqrt{10}$ and within the upper half sphere $r^2 + z^2 = 49$.

The volume of the cone consists of a pile of disks of variable radius $r = z/3$ as

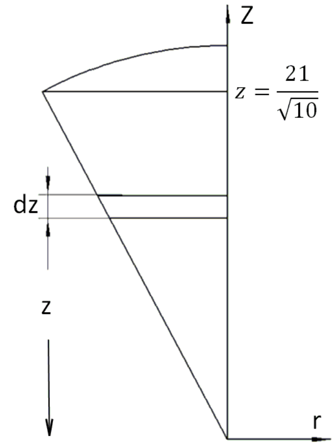

Fig. 5. Line integrals of a crown and a flat cone: Symmetric and side view of the upper half cone within the sphere in a cylindrical coordinate system

prescribed by the cone $z = 3r$. The value of V_1 is evaluated in a line integral as:

$$\begin{aligned} V_1 &= \int_0^{\frac{21}{\sqrt{10}}} \pi r^2 \, dz \\ &= \int_0^{\frac{21}{\sqrt{10}}} \pi \frac{z^2}{9} \, dz \\ &= \frac{\pi}{10\sqrt{10}} \, 7^3. \end{aligned} \tag{16}$$

The volume of the crown consists of a pile of disks of variable radius r given by the upper half sphere $r^2 + z^2 = 49$. However, these disks are piled up from the plane $z = 21/\sqrt{10}$ to the tip of the upper half sphere. The value of V_2 is then found with another line integral as below:

$$\begin{aligned} V_2 &= \int_{\frac{21}{\sqrt{10}}}^{7} \pi r^2 \, dz \\ &= \int_{\frac{21}{\sqrt{10}}}^{7} \pi (49 - z^2) \, dz \\ &= \frac{2\pi}{3} \, 7^3 - \frac{21\pi}{10\sqrt{10}} \, 7^3. \end{aligned} \tag{17}$$

Therefore, the volume of V_0 is the sum of the above two:

$$\begin{aligned} V_0 &= V_1 + V_2 \\ &= \frac{\pi}{10\sqrt{10}} \, 7^3 + \frac{2\pi}{3} \, 7^3 - \frac{21\pi}{10\sqrt{10}} \, 7^3 \\ &= \frac{2\pi}{3} \, 7^3 - \frac{2\pi}{\sqrt{10}} \, 7^3. \end{aligned} \tag{18}$$

Although by common sense, the volume of the upper half sphere V_s is known, it can be easily found via below line integral:

$$\begin{aligned} V_s &= \int_0^7 \pi r^2 \, dz \\ &= \int_0^7 \pi (49 - z^2) \, dz \\ &= \pi \, 7^3 - \frac{\pi}{3} \, 7^3 \\ &= \frac{2\pi}{3} \, 7^3. \end{aligned} \tag{19}$$

Finally, the desired volume V is the difference of V_s and V_0:

$$V = V_s - V_0 = \frac{2\pi}{\sqrt{10}} \, 7^3. \tag{20}$$

The last approach is a bit tedious but only consists of very simple line integrals.

4 Conclusion

This paper summarizes some strategies for exact evaluation of some triple integrals, presents several alternative perspectives for setting up triple integrals defined over complex domains, and provides examples of assessing triple integrals with diverse coordinate systems.

These different approaches could provide college students cross-checks for evaluating triple integrals on complex domains, stimulate enthusiasm and curiosity of them to inspire more thoughts afterward, and hopefully help assist them in evaluating triple integral over complex domains.

Acknowledgements

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References

[1] H. Kestelman, Modern Theories of Integration, 2^{nd} Rev. Ed., Dover Publications, Inc., 1960.

[2] H. Jeffreys, B. Jeffreys, Methods of Mathematical Physics, Cambridge University Press, 1988.

[3] W. Briggs, L. Cochran, B. Gillett, Calculus Early Transcendentals, Pearson, 2011.

[4] J. Rogawski, C. Adams, Calculus Early Transcendentals 3^{rd} Ed., W. H. Freeman and Company, 2015.

[5] J. Stewart, Calculus Early Transcendentals 7^{th} Ed., Brook/Cole, Inc., 2012.