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Abstract

The bosonic representation of the comma ghost in the full string basis is examined in full. The proof that the comma 3- vertex (matter and ghost) in the bosonic representation satisfy the Ward like identities is established thus completing the proof of the Bose Fermi equivalence in the comma theory.

Contents

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1 Introduction

The recent work of references [1, 2, 3, 4] has generated much interest in the comma formulation of Witten's theory of interacting open bosonic strings. An important role in the formulation of the comma theory is played by the $BRST$ charge Q of the first quantized theory [5, 6]. In general, the $BRST$ invariance of the first quantized theory becomes a gauge invariance of the second theory [7, 8, 9, 10]. In the interacting comma theory, the role of the ghost fields is quite subtle. The rich structure of the ghost sector of the interacting theory deserves more consideration. The comma theory is intimately related to the conformally invariant $2-$dimensional field theory. It is possible to study the ghosts using either bosons or fermions. However, to relate the fermionic ghost formulation to the bosonic ghost formulation, we have to carry a bosonization procedure explicitly [9, 11, 12, 13]. The bosonic and the fermionic realization of the ghost fields for the comma theory have been used in ref. [14] and [5] respectively, to write down the ghost part of the three-comma vertex and the proof of equivalence was only addressed partially in [13]. Although both formulations give a gauge invariant theory, their equivalence is not at all transparent. It is the purpose of this paper to complete the proof of equivalence for both formulations. The key to the proof as we have seen in the first part of the proof [13] lies on the various identities satisfied by the G-coefficients that define the comma interacting vertex. To complete the proof of equivalence in [13], we have to show that both realizations of the comma vertex satisfy the same Ward-like identities.

Like in the matter sector for the comma formulation of Witten's theory for open bosonic strings, in the half-string approach to the ghost part of string theory, the elements of the theory are defined by $\delta - function$ type overlaps

$$V_3^{\phi} = \exp\left(i \sum_{r=1}^{3} Q_r^{\phi} \phi\left(\pi/2\right) \right) V_{3,0}^{\phi} \tag{1}$$

where

$$V_{3,0}^{\phi} = \prod_{r=1}^{3} \prod_{\sigma=0}^{\pi/2} \delta\left(\phi_r^L\left(\sigma\right) - \phi_{r-1}^R\left(\sigma\right) \right) \tag{2}$$

and the half string ghost coordinates defined in [14, 13], $\phi_r^{L,R}\left(\sigma\right)$, $r = 1, 2, 3$, are given by

$$\begin{aligned} \phi_r^L\left(\sigma\right) &= \phi_r(\sigma) - \phi_r(\frac{\pi}{2}), \quad \sigma \in \left[0, \frac{\pi}{2}\right] \\ \phi_r^R\left(\sigma\right) &= \phi_r(\pi - \sigma) - \phi_r(\frac{\pi}{2}), \quad \sigma \in \left[0, \frac{\pi}{2}\right] \end{aligned} \tag{3}$$

where $\phi_r(\sigma)$ is the full string coordinate

$$\phi_r\left(\sigma\right) = \phi_0^r + \sqrt{2} \sum_{n=1}^{\infty} \phi_n^r \cos n\sigma, \quad \sigma \in [0, \pi] \tag{4}$$

The index r refers to the rth string (it is to be understood that $r-1=0\equiv 3$). The factor Q_r^{ϕ} is the ghost number insertion at the mid-point which is needed for the $BRST$ invariance of the theory [16, 14] and in this case $Q_1^{\phi}=Q_2^{\phi}=Q_1^3=1/2$. As we have seen before in the Hilbert space of the theory, the $\delta-functions$ translate into operator overlap equations which determine the precise form of the vertex. The ghost part of the comma vertex in the full string basis has the same structure as the coordinate one apart from the mid-point insertions

$$|V_{\phi}^{HS}>=e^{\frac{1}{2}i\sum_{r=1}^{3}\phi^{r}(\pi/2)}V_{\phi}^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)\left|0,N_{ghost}=\frac{3}{2}\right)_{123}^{\phi} \tag{5}$$

where the α's are the bosonic oscillators defined by the expansion of the bosonized ghost $\left(\phi\left(\sigma\right),p^{\phi}\left(\sigma\right)\right)$ fields and $V_{\phi}^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)$ is the exponential of the quadratic form in the ghost creation operators with the same structure as the coordinate piece of the vertex

$$\begin{aligned} V_{\phi}^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right) &= \exp\Bigg[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m=1}^{\infty}\alpha_{-n}^{\phi,r}G_{nm}^{rs}\alpha_{-m}^{\phi,s}+ \\ &\sum_{r,s=1}^{3}p_0^{\phi,r}G_{0m}^{rs}\alpha_{-m}^{\phi,s}+\frac{1}{2}\sum_{r,s=1}^{3}p^{\phi,r}G_{00}^{rs}p_0^{\phi,s}\Bigg] \end{aligned} \tag{6}$$

where the the matrix elements G_{nm}^{rs} have been constructed in [15].

In the full string, the fermionic ghost overlap equations are

$$\begin{aligned} c_r\left(\sigma\right) &= c_{r-1}(\pi-\sigma) \quad \sigma\in\left[0,\frac{\pi}{2}\right] \\ c_r\left(\sigma\right) &= -c_{r+1}(\pi-\sigma)\ ,\quad \sigma\in\left[\frac{\pi}{2},\pi\right] \end{aligned} \tag{7}$$

and

$$\begin{aligned} b_r\left(\sigma\right) &= b_{r-1}(\pi-\sigma) \quad \sigma\in\left[0,\frac{\pi}{2}\right] \\ b_r\left(\sigma\right) &= b_{r+1}(\pi-\sigma)\ ,\quad \sigma\in\left[\frac{\pi}{2},\pi\right] \end{aligned} \tag{8}$$

The proof of the bose-fermi equivalence involves two major obstacles. The first is to show that the bosonized half string ghosts, (5), satisfy the $c-$ and $b-overlap$ equations displayed above. To carry out the proof, in [13], the authors utilized the bosonization formulas

$$c_+\left(\sigma\right)=:e^{i\phi_+(\sigma)}:,\qquad b_+\left(\sigma\right)=:e^{-i\phi_-(\sigma)}: \tag{9}$$

where $\phi\left(\sigma\right)=\frac{1}{2}\left(\phi_+\left(\sigma\right)+\phi_-\left(\sigma\right)\right)$, and

$$\phi_{\pm}\left(\sigma\right)=\phi_0\pm\sigma\left(p_0^{\phi}+\frac{1}{2}\right)+i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_n^{\phi}e^{\mp in\sigma}-\alpha_{-n}^{\phi}e^{\pm in\sigma}\right) \tag{10}$$

The fermionic ghost coordinates of the bosonic string are anticommuting fields

$$
\begin{aligned}
c_{\pm}(\sigma) &= c(\sigma) \pm i\pi_b(\sigma) \quad \sigma \in [0,\pi] \\
b\pm(\sigma) &= \pi_c(\sigma) \pm ib(\sigma) \quad , \quad \sigma \in [0,\pi]
\end{aligned}
\tag{11}
$$

The $c_+\,(c_-)$ are the ghosts for reparametrization of $z = \tau + i\sigma$ $(\overline{z} = \tau - i\sigma)$ respectively and the $b_{\pm}$ are the corresponding anti-ghosts. These obey the anticommutation relations

$$
\begin{aligned}
\{c_n, c_m\} &= \{b_n, b_m\} = 0 \\
\{c_n, b_m\} &= \delta_{n+m\ 0}
\end{aligned}
\tag{12}
$$

The fermionic half string ghosts were defined in [5] by

$$
\begin{aligned}
c_r^L(\sigma) &= c_r(\sigma) - c_r(\frac{\pi}{2}), \quad \sigma \in \left[0, \frac{\pi}{2}\right] \\
c_r^R(\sigma) &= c_r(\pi - \sigma) - c_r(\frac{\pi}{2}), \quad \sigma \in \left[0, \frac{\pi}{2}\right]
\end{aligned}
\tag{13}
$$

and likewise for $b^L(\sigma)$ and $b^R(\sigma)$.

2 The proof of the Ward-like Identities

The bosonization of the fermionic coordinates, using the standard procedure (see Ref. 17 in [8]), it is not obvious that all ingredients of the theory employing the bosonic field $\phi^L(\sigma)$ and $\phi^R(\sigma)$ are equivalent to those constructed using the original Fermi fields appearing in the left hand side of the above relations. It has been shown in [13], that the ghost vertices in the half string operator formulation obey the same overlap equations as the fermionic vertices and are identical. Consequently one is free to use either formulation of the ghost sector of the theory. In fact this statement is only partially true in [13], the authors failed to establish that the half string ghost (plus matter) vertex in the bosonic realization of the ghosts satisfy the same Ward-like identities obeyed by the half string ghost (plus matter) vertex in the fermionic realization of the ghosts. To complete the equivalence between the two realization of the half string ghost vertex, we need to establish the Ward-like identities utilizing the bosonic representation of the half string ghost as well.

The Ward-like identities for the Witten vertex matter plus ghost [12] in the fermionic representation are given by

$$
W_m^{x+c,r} | V_W^{x+c} >= 0, \quad m = 1, 2, ... \tag{14}
$$

where $W_m^{x+c,r}$; henceforth refered to as the Ward-like operator, is defined by

$$
W_m^{x+c,r} = W_m^{x,r} + W_m^{c,r} = L_m^{x+c,r} + \sum_{s=1}^{3} \sum_{n=0}^{\infty} m \widetilde{N}_{mn}^{rs} L_{-n}^{x+c,s} \tag{15}
$$

and

$$|V_W^{x+c}>=|V_W^x>|V_W^c> \tag{16}$$

The Virasoro generators for both matter and ghost coordinates are given by

$$\begin{aligned} L_m^{x,r} &= \sum_{k=1}^{\infty} \alpha_{-k}^r \alpha_{k+m}^r + \frac{1}{2} \sum_{k=1}^{m-1} \alpha_{m-k}^r \alpha_k^r + p_0^r \alpha_m^r & (17) \\ L_0^{x,r} &= \frac{1}{2} (p_0^r)^2 + \sum_{k=1}^{\infty} \alpha_{-k}^r \alpha_k^r, & (18) \end{aligned}$$

and

$$\begin{aligned} L_m^{c,r} &= \sum_{k=1}^{\infty} \left[(2m+k)\, b_{-k}^r c_{k+m}^r - (m-k)\, c_{-k}^r b_{k+m}^r \right] \\ &\quad + \sum_{k=1}^{m-1} (m+k)\, b_{m-k}^r c_k^r + m b_m^r c_0^r - 2m b_0^r c_m^r & (19) \\ L_0^{c,r} &= \sum_{k=1}^{\infty} k \left(b_{-k}^r c_k^r - c_{-k}^r b_k^r \right), & (20) \end{aligned}$$

respectively. Here m takes integral values greater than 0. If the half string bosonic ghost version of the vertex is equivalent to the fermionic version then it must obey an identity of the form (14) and the anomaly of the ghost part must cancel the anomaly of the coordinate. Our job is thus to show that the half string full vertex (i.e., matter plus ghost) in the bosonic representation satisfies:

$$W_m^{x+\phi,r}|V_H^{x+\phi}>=0, \quad m=1,2,... \tag{21}$$

where the Ward-like operator in this case is expressed in the bosonic representation of the ghost

$$W_m^{x+\phi,r} = W_m^{x,r} + W_m^{\phi,r} = L_m^{x+\phi,r} + \sum_{s=1}^{3} \sum_{n=0}^{\infty} m \widetilde{N}_{mn}^{rs} L_{-n}^{x+\phi,s} \tag{22}$$

As before in the above expression $L_n^{x+\phi} = L_n^x + L_n^\phi$ and $|V_H^{x+\phi}>=|V_H^x>|V_H^\phi>$. The Virasoro generators for the ghost in terms of the bosonic operators are given by

$$\begin{aligned} L_m^{\phi,r} &= \sum_{k=1}^{\infty} \alpha_{-k}^{\phi,r} \alpha_{k+m}^{\phi,r} + \frac{1}{2} \sum_{k=1}^{m-1} \alpha_{m-k}^{\phi,r} \alpha_k^{\phi,r} + \left(p_0^{\phi,r} - \frac{3}{2} m \right) \alpha_m^{\phi,r}, & (23) \\ L_0^{\phi,r} &= \frac{1}{2} \left(p_0^{\phi,r} \right)^2 - \frac{1}{8} + \sum_{k=1}^{\infty} \alpha_{-k}^{\phi,r} \alpha_k^{\phi,r}, & (24) \end{aligned}$$

where $m > 0$. The extra term in (23), that is, the linear term in $\alpha_m^{\phi,r}$ arises because of the presence of the $R\phi$ term in the action [9] of the bosonized ghosts

$$I_\phi = \frac{1}{2\pi}\int d^2\sigma\left(\partial_\beta\phi\partial^\beta\phi - 3iR\phi\right) \tag{25}$$

or alternatively because of the extra linear term in the ghost energy-momentum tensor

$$T_\phi = \frac{1}{2\pi}\left[\left(\partial_\pm\phi\right)^2 - \frac{3}{2}\partial_\pm\phi\right] \tag{26}$$

The extra term is needed [9, 16, 14] and must have precisely the coefficient given in (23), so that ϕ can cancel the Virasoro anomaly of the x^μ so that the total Fourier components of the energy momentum

$$L_m^r = L_m^{x,r} + L_m^{\phi,r} - \frac{9}{8}\delta_{n0} \tag{27}$$

obey the Virasoro algebra

$$\left[L_m^r, L_n^s\right] = (m-n)\,L_{m+n}^r \tag{28}$$

which is free of central charge.

We will show that the comma vertex $|V_{HS}^{x+\phi}>=|V_{HS}^{x}>|V_{HS}^{\phi}>$ indeed satisfies the Ward-like identities stated in equation (21). It is more convenient to recast the comma three-point vertex in the full string oscillator basis [15].

To express the comma vertex in an exponential form in the creation operators only we need to commute the annihilation part of the ghost mid-point insertion $\frac{1}{2}i\sum\phi^r\left(\pi/2\right)$ in (5) through the creation part of the vertex. The normal mode expansion of the ghost field $\phi\left(\sigma\right)$ is at $\tau = 0$

$$\phi^r\left(\sigma\right) = \phi_0^r + \sqrt{2}\sum_{n=1}^{\infty}\phi_n^r\cos n\sigma \tag{29}$$

The mid-point of the ghost coordinate is obtained by substituting $\sigma = \pi/2$ in the above expression

$$\phi^r\left(\frac{\pi}{2}\right) = \phi_0^r + i\sum_{n=1}^{\infty}\lambda_n\left(\alpha_n^{\phi,r} - \alpha_{-n}^{\phi,r}\right) \tag{30}$$

where $\lambda_n = (n)^{-1}\cos\left(n\pi/2\right)$, $n = 1, 2, 3, ...$ Let us first consider the first factor in (5)

$$\begin{aligned}\exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi^r\left(\pi/2\right)\right) &= N_1\exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi_0^r\right) \\ &\times\exp\left(\frac{1}{2}\sum_{r=1}^{3}\sum_{n=1}^{\infty}\lambda_n\alpha_{-n}^{\phi,r}\right)\exp\left(-\frac{1}{2}\sum_{r=1}^{3}\sum_{n=1}^{\infty}\lambda_n\alpha_n^{\phi,r}\right)\end{aligned} \tag{31}$$

where $N_1 = \exp\left[-3 \cdot 2^3 \sum_{n=1}^{\infty} n\lambda_n\right] = -3 \cdot 2^3 \, \zeta(1)$. In obtaining this result we made use of the well known identity

$$e^{\hat{A}_1} e^{\hat{A}_2} = e^{\frac{1}{2}\left[\hat{A}_1, \hat{A}_2\right]} e^{\hat{A}_1 + \hat{A}_2} \tag{32}$$

which is valid when $\left[\hat{A}_1, \hat{A}_2\right]$ is a C number. We notice that the C-number is $-3/8$ times the Riemann zeta function $\zeta(1)$. The next step is to bring the annihilation part of the mid-point insertions to act on the ghost vacuum. So we need to consider the operator product

$$\exp\left(-\frac{1}{2}\sum_{r=1}^{3}\sum_{n=0}^{\infty}\widetilde{\lambda}_n \alpha_n^{\phi,r}\right) V_\phi^{HS}\left(\alpha^{\phi,1\dagger}, \alpha^{\phi,2\dagger}, \alpha^{\phi,3\dagger}\right) \tag{33}$$

where $\widetilde{\lambda}_n \equiv \lambda_n$ for $n > 0$ and $\widetilde{\lambda}_0 = 0$. To commute the annihilation operators $\alpha_n^{\phi,r}$ through the creation operators $a_{-n}^{\phi,s}$, $a_{-m}^{\phi,s}$, we note that the exponential of the quadratic form is the Gaussian

$$\begin{aligned} V_\phi^{HS}\left(\alpha^{\phi,1\dagger}, \alpha^{\phi,2\dagger}, \alpha^{\phi,3\dagger}\right) &= \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m=0}^{\infty} a_{-n}^{\phi,r} G_{nm}^{rs} a_{-m}^{\phi,s}\right) \\ &= \lim_{N\to\infty} \pi^{-N/2}\left[\det G\right]^{-1/2} \int Dx \exp\left(-\frac{1}{2}\overrightarrow{x}^T G^{-1} \overrightarrow{x}\right) \\ &\quad \times \exp\left(\sum_{s=1}^{3}\sum_{m=0}^{\infty} a_{-m}^{\phi,s} x_m^s\right) \end{aligned} \tag{34}$$

where $Dx = \prod_{r=1}^{3}\prod_{n=0}^{N} dx_n^r$. Thus we first need to consider

$$\exp\left(-\frac{1}{2}\sum_{r=1}^{3}\sum_{n=0}^{\infty}\widetilde{\lambda}_n \alpha_n^{\phi,r}\right) \exp\left(\sum_{s=1}^{3}\sum_{m=0}^{\infty} a_{-m}^{\phi,s} x_m^s\right) \tag{35}$$

Using the well known identity

$$e^{\hat{A}_1} e^{\hat{A}_2} = e^{\left[\hat{A}_1, \hat{A}_2\right]} e^{\hat{A}_2} e^{\hat{A}_1} \tag{36}$$

which is valid when $\left[\hat{A}_1, \hat{A}_2\right]$ is a C number, the above product becomes

$$\begin{aligned} &\exp\left(-\frac{1}{2}\sum_{r=1}^{3}\sum_{n=0}^{\infty}\widetilde{\lambda}_n \alpha_n^{\phi,r}\right) \exp\left(\sum_{s=1}^{3}\sum_{m=0}^{\infty} a_{-m}^{\phi,s} x_m^s\right) \\ = &\exp\left[\sum_{s=1}^{3}\sum_{m=0}^{\infty}\left(-\frac{1}{2}m\widetilde{\lambda}_m + a_{-m}^{\phi,s}\right) x_m^s\right] \exp\left(-\frac{1}{2}\sum_{r=1}^{3}\sum_{n=0}^{\infty}\widetilde{\lambda}_n \alpha_n^{\phi,r}\right) \end{aligned} \tag{37}$$

Observe that the operator $\overrightarrow{\alpha}$ translates $\overrightarrow{\alpha}^\dagger$ by $-m\widetilde{\lambda}_m/2$. With the help of the identities in (34) and (37), the operator product in (33) becomes

$$\exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m=0}^{\infty}\left(-\frac{1}{2}n\widetilde{\lambda}_n+a_{-n}^{\phi,r}\right)G_{nm}^{rs}\left(-\frac{1}{2}m\widetilde{\lambda}_m+a_{-m}^{\phi,s}\right)\right]$$
$$\times\exp\left(-\frac{1}{2}\sum_{r=1}^{3}\sum_{n=1}^{\infty}\widetilde{\lambda}_n\alpha_n^{\phi,r}\right) \tag{38}$$

Since $\alpha_n^{\phi,r}|0,0,0)=0$, the above expression when acting on the vacuum of the three strings gives

$$\exp\left(-\frac{1}{2}\sum_{r=1}^{3}\sum_{n=0}^{\infty}\widetilde{\lambda}_n\alpha_n^{\phi,r}\right)V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0,0,0>_\phi$$
$$=\exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m=0}^{\infty}\left(-\frac{1}{2}n\widetilde{\lambda}_n+a_{-n}^{\phi,r}\right)G_{nm}^{rs}\left(-\frac{1}{2}m\widetilde{\lambda}_m+a_{-m}^{\phi,s}\right)\right]|0,0,0>_\phi \tag{39}$$

Using the identity

$$\sum_{r=1}^{3}G_{nm}^{rs}=\frac{(-1)^{n+1}}{n}\delta_{nm} \tag{40}$$

and the definition of $\widetilde{\lambda}$, the above expression becomes

$$\exp\left(-\frac{1}{2}\sum_{r=1}^{3}\sum_{n=0}^{\infty}\widetilde{\lambda}_n\alpha_n^{\phi,r}\right)V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0,0,0>_\phi$$
$$=N_2\exp\left[\sum_{s=1}^{3}\sum_{m=1}^{\infty}\lambda_m^s a_{-m}^{\phi,s}\right]V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0,0,0>_\phi \tag{41}$$

where $N_2=\exp\left[2^{-3}\sum_{r,s=1}^{3}\sum_{n,m=1}^{\infty}\cos(n\pi/2)\,G_{nm}^{rs}\cos(m\pi/2)\right]$. Combining equation (41), (31), and (5), and replacing λ_n by $(1/n)\cos(n\pi/2)$, we find

$$|V_\phi^{HS}>=N\exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi_0^r\right)\exp\left(\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right)$$
$$\times V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0,0,0>_\phi \tag{42}$$

where $N=N_1N_2$ is a constant that can be absorbed in the overall normalization constant. We note that the net effect of commuting the annihilation operator piece of the mid-point insertions $i\sum_{r=1}^{3}\phi^r(\pi/2)/2$ is doubling the coefficient of creation operator piece in the mid-point insertions.

Before we could apply the Virasoro to the vertex we need first to commute the annihilation operators in the Virasoro generators through the ghost insertions. Thus we need to consider

$$\alpha_n^{\phi,r}\exp\left(\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right),\ n=1,2,3,... \tag{43}$$

If we write $\alpha_n^{\phi,r}$ as $\alpha_n^{\phi,r} = (\partial/\partial\rho_n^r)\exp\left(\sum_{s=1}^3\sum_{m=0}^\infty \rho_m^s\alpha_m^s\right)|_{\vec{\rho}=0}$ and use the operator identity in (32), then the above expression becomes

$$\alpha_n^{\phi,r}\exp\left(\sum_{r=1}^3\sum_{n=1}^\infty \frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right) = \exp\left(\sum_{r=1}^3\sum_{n=1}^\infty \frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right) \times\left(\alpha_n^{\phi r}+\cos\frac{n\pi}{2}\right) \tag{44}$$

where $n = 1, 2, 3, ...$ For $p_0^{\phi,r}$, we have

$$p_0^{\phi,r}\exp\left(\frac{1}{2}i\sum_{s=1}^3\phi_0^r\right) = \exp\left(\frac{1}{2}i\sum_{s=1}^3\phi_0^s\right)\left(p_0^{\phi,r}+\frac{1}{2}\right) \tag{45}$$

Thus we see that the effect of commuting the annihilation operators in the Virasoro generators through the ghost insertions amounts to a shift of the annihilation operator in the Virasoro generator by

$$\alpha_n^{\phi,r} \to \left(\alpha_n^{\phi,r}+\cos\frac{n\pi}{2}\right) \tag{46}$$

for $n = 1, 2, 3, ...,$ and

$$p_0^{\phi,r} \to \left(p_0^{\phi,r}+\frac{1}{2}\right) \tag{47}$$

Notice that this shift is independent of the string index r.

To commute the annihilation part of the α operators in the Virasoro generators L, we need to commute α_n^r through the creation part of the vertex; that is; we need

$$\alpha_k^t V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right) \tag{48}$$

The above expression may be written as

$$\begin{aligned} &\alpha_k^t V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right) \qquad (49)\\ =\;& \alpha_k^t\exp\left(\frac{1}{2}\sum_{r,s=1}^3\sum_{n,m=0}^\infty a_{-n}^{\phi,r}G_{nm}^{rs}a_{-m}^{\phi,s}\right)\\ =\;& \frac{\partial}{\partial\rho_k^t}\left\{\exp\left(\sum_{s=1}^3\sum_{m=0}^\infty\rho_m^s\alpha_m^{\phi,s}\right)\exp\left(\frac{1}{2}\sum_{r,s=1}^3\sum_{n,m=0}^\infty a_{-n}^{\phi,r}G_{nm}^{rs}a_{-m}^{\phi,s}\right)\right\}|_{\vec{\rho}=0}\end{aligned}$$

The expression inside the curly brackets is identical to equation (33) with ρ_m^s replacing $-\frac{1}{2}\widetilde{\lambda}_n$; thus the result can be obtained from (38) with ρ_m^s replacing $-\frac{1}{2}\widetilde{\lambda}_n$; hence the above expression becomes

$$\alpha_k^t V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right) = \frac{\partial}{\partial\rho_k^t}\exp\left[\frac{1}{2}\sum_{r,s=1}^3\sum_{n,m=0}^\infty\left((n+\delta_{n0})\rho_n^r + a_{-n}^{\phi,r}\right)\right.$$
$$\left.G_{nm}^{rs}\left((m+\delta_{m0})\rho_m^s+a_{-m}^{\phi,s}\right)\right]\exp\left(\sum_{s=1}^3\sum_{m=0}^\infty\rho_m^s\alpha_m^{\phi,s}\right)|_{\vec{\rho}=0} \tag{50}$$

So we have succeeded in commuting the annihilation operator $\alpha_n^{\phi,r}$ through the creation part of the vertex. Since $\alpha_n^{\phi,r}$ annihilates the ghost part of the vacuum of the three strings, then

$$\exp\left(\sum_{s=1}^{3}\sum_{m=0}^{\infty}\rho_m^s\alpha_m^{\phi,s}\right)|_{\vec{\rho}=0}|0>_{123}^{\phi}=|0>_{123}^{\phi}$$

and equation (50) becomes

$$\begin{aligned}&\alpha_n^r V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0>_{123}^{\phi}\\ =\;&(n+\delta_{n0})\sum_{s=1}^{3}\sum_{m=0}^{\infty}G_{nm}^{rs}a_{-m}^{\phi,s}V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0>_{123}^{\phi}\end{aligned}\quad(51)$$

where $n,m=0,1,2,3,....$ In fact this relation is equivalent to the ϕ and p_ϕ overlaps on $V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0>_{123}^{\phi}$. This identity will be very useful as we shall see shortly in commuting all the annihilation operators $\alpha^{\phi\prime}$s in the Virasoro generators L_n through $V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)$.

Let us first commute $L_m^{\phi,r}$ through the ghost insertions. Using the identities in (46) and (47), we find

$$\begin{aligned}&L_m^{\phi,r}\exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi_0^r\right)\exp\left(\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right)\\ =\;&\exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi_0^r\right)\exp\left(\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right)\left\{\left[L_m^{\phi,r}+\frac{3}{2}m\alpha_m^{\phi,r}\right]\right.\\ &+\left[\sum_{k=1}^{\infty}\alpha_{-k}^{\phi,r}\cos\pi\frac{k+m}{2}+\sum_{k=1}^{m-1}\alpha_k^{r,\phi}\cos\pi\frac{m-k}{2}+p_0^{\phi,r}\cos\frac{m\pi}{2}+\left(\frac{1}{2}-\right.\right.\\ &\left.\left.\frac{3m}{2}\right)\alpha_m^{r,\phi}\right]+\left.\left[\frac{1}{2}\sum_{k=1}^{m-1}\cos\pi\frac{m-k}{2}\cos\frac{k\pi}{2}+\left(\frac{1}{2}-\frac{3m}{2}\right)\cos\frac{m\pi}{2}\right]\right\}\end{aligned}\quad(52)$$

where $m=1,2,....$ In obtaining the above result we made use of the fact that $\sum_{k=1}^{m-1}\alpha_{m-k}^{\phi,r}\cos\frac{k\pi}{2}=\frac{1}{2}\sum_{k=1}^{m-1}\alpha_k^{r,\phi}\cos\pi\frac{m-k}{2}$. Observe that the quadratic part[1] (i.e., the expression in the first square bracket $L_m^{\phi,r}+\frac{3}{2}m\alpha_m^{\phi,r}$) is identical to Virasoro generator for the orbital part $L_m^{x,r}$. Thus its action on $V_\phi^{HS}|0>_{123}^{\phi}$ is identical to the action of $L_m^{x,r}$ on the coordinate part of the vertex because V_ϕ^{HS} and V_x^{HS} has exactly the same structure. The expression in the second square brackets is linear in oscillators and the expression in the third square brackets has no oscillators. We still need to compute the effect of passing the expression in the second brackets through the exponential of the quadratic form in the

[1] Note that the expression $L_{-m}^{\phi,r}+\frac{3}{2}m\alpha_{-m}^{\phi,r}$ is indeed quadratic in the creation-annihilation operators $\alpha^{\phi}$and $\alpha^{\phi\dagger}$ since the linear term $\frac{3}{2}m\alpha_{-m}^{\phi,r}$ cancels agains the linear term in $L_{-m}^{\phi,r}$.

ghost creation operators. But before we do that let us first see the effect of passing $L_{-m}^{\phi,r}$ through the mid-point insertions. Since $L_{-m}^{\phi,r} \equiv L_{m}^{\phi,r\dagger}$, then taking the adjoint of (23), we find

$$\begin{aligned} & L_{-m}^{\phi,r} \exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi_0^r\right)\exp\left(\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right) \\ = \;& \exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi_0^r\right)\exp\left(\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right)\left\{\left[L_{-m}^{\phi,r}+\frac{3}{2}m\alpha_{-m}^{\phi,r}\right]+\right. \\ & \left.\left[\sum_{k=1}^{\infty}\alpha_{-k-m}^{\phi,r}\cos\frac{k\pi}{2}+\left(\frac{1}{2}-\frac{3}{2}m\right)\alpha_{-m}^{\phi,r}\right]\right\} \end{aligned} \tag{53}$$

where $m = 1, 2, ...$ Once more the quadratic[2] inside the first square bracket; that is, $L_{-m}^{\phi,r} + \frac{3}{2}m\alpha_{-m}^{\phi,r}$ is identical to the orbital part of the Virasoro generator $L_{-m}^{x,r}$ and so its effect on the exponential of the quadratic form in the ghost creation operators is the same as the action of $L_{-m}^{x,r}$ on the orbital part of the vertex. The effect of the expression in the second square brackets needs to be computed which we shall do later. For the zero mode of the ghost Virasoro generators, we find

$$\begin{aligned} & L_{0}^{\phi,r} \exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi_0^r\right)\exp\left(\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right) \\ = \;& \exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi_0^r\right)\exp\left(\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right)\left\{\left[L_{0}^{\phi,r}+\frac{1}{8}\right]\right. \\ & \left.+\left[\frac{1}{2}p_0^{\phi,r}+\sum_{k=1}^{\infty}\cos\frac{k\pi}{2}\alpha_{-k}^{\phi,r}\right]+\left[\frac{1}{2}\cdot\frac{1}{2^2}-\frac{1}{8}\right]\right\} \end{aligned} \tag{54}$$

Once again the expression inside the first square brackets is identical to that for the orbital zero mode of the Virasoro generator $L_0^{x,r}$; hence its action is the same as that of $L_0^{x,r}$ on the orbital part of the vertex. The second and the third expressions in the second and the third square brackets have no similar terms in the orbital part $L_0^{x,r}$. Using equations (52), (53), and (54) to commute the ghost part of the Ward operator through the mid-point insertions, we find

$$\begin{aligned} & W_{m}^{\phi,r} \exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi_0^r\right)\exp\left(\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right) \\ = \;& \exp\left(\frac{1}{2}i\sum_{r=1}^{3}\phi_0^r\right)\exp\left(\sum_{r=1}^{3}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\alpha_{-n}^{\phi,r}\right) \\ & \left[W_m^{\phi,r}(2)+W_m^{\phi,r}(1)+\kappa_m^{\phi,r}(1)\right] \end{aligned} \tag{55}$$

[2] See previous footnote.

where

$$
\begin{aligned}
W_m^{\phi,r}(2) \quad &\equiv \quad \sum_{k=1}^{\infty} \alpha_{-k}^{\phi,r} \alpha_{k+m}^{r,\phi} + \frac{1}{2} \sum_{k=1}^{m-1} \alpha_{m-k}^{\phi,r} \alpha_k^{r,\phi} + p_0^{\phi,r} \alpha_m^{r,\phi} \\
&+ \sum_{s=1}^{3} \sum_{n=1}^{\infty} m \widetilde{N}_{mn}^{rs} \left[\sum_{k=1}^{\infty} \alpha_{-k-n}^{\phi,s} \alpha_k^{s,\phi} + \frac{1}{2} \sum_{k=1}^{m-1} \alpha_{-k}^{\phi,s} \alpha_{-n+k}^{\phi,s} + p_0^{\phi,s} \alpha_{-n}^{\phi,s} \right] \\
&+ \sum_{s=1}^{3} m \widetilde{N}_{m0}^{rs} \left[\frac{1}{2} \left(p_0^{\phi,s} \right)^2 + \sum_{k=1}^{\infty} \alpha_{-k}^{\phi,s} \alpha_k^{\phi,s} \right] \\
&= \quad \left[L_m^{\phi,r} + \frac{3}{2} m \alpha_m^{\phi,r} \right] + \sum_{s=1}^{3} \sum_{n=1}^{\infty} m \widetilde{N}_{mn}^{rs} \left[L_{-n}^{\phi,r} + \frac{3}{2} n \alpha_{-n}^{\phi,r} \right] + \sum_{s=1}^{3} m \widetilde{N}_{m0}^{rs} \left[L_0^{\phi,s} + \frac{1}{8} \right]
\end{aligned}
$$

$$
\begin{aligned}
W_m^{\phi,r}(1) \quad &\equiv \quad \left[\sum_{k=1}^{\infty} \alpha_{-k}^{\phi,r} \cos \pi \frac{k+m}{2} + \sum_{k=1}^{m-1} \alpha_k^{r,\phi} \cos \pi \frac{m-k}{2} + p_0^{\phi,r} \cos \frac{m\pi}{2} + \left(\frac{1}{2} - \frac{3}{2} m \right) \alpha_m^{r,\phi} \right] \\
&+ \sum_{s=1}^{3} \sum_{n=1}^{\infty} m \widetilde{N}_{mn}^{rs} \left[\sum_{k=1}^{\infty} \alpha_{-k-n}^{\phi,s} \cos \frac{k\pi}{2} + \left(\frac{1}{2} - \frac{3}{2} n \right) \alpha_{-n}^{\phi,s} \right] \\
&+ \sum_{s=1}^{3} m \widetilde{N}_{m0}^{rs} \left[\frac{1}{2} p_0^{\phi,s} + \sum_{k=1}^{\infty} \cos \frac{k\pi}{2} \alpha_{-k}^{\phi,s} \right],
\end{aligned} \tag{56}
$$

and

$$
\begin{aligned}
\kappa_m^{\phi,r}(1) \quad &\equiv \quad \left[\frac{1}{2} \sum_{k=1}^{m-1} \cos \pi \frac{m-k}{2} \cos \frac{k\pi}{2} + \left(\frac{1}{2} - \frac{3}{2} m \right) \cos \frac{m\pi}{2} \right] + \left[\frac{1}{2} \cdot \frac{1}{2^2} - \frac{1}{8} \right] \\
&= \quad \frac{1 + (-1)^m}{2} \frac{5}{2} \frac{m}{2} (-1)^{m/2}
\end{aligned} \tag{57}
$$

It is important to notice that $W_m^{\phi,r}(2)$ has precisely the same structure as the orbital part $W_m^{x,r}$ of the Ward operator. Thus its action on $V_\phi^{HS}\left(\alpha^{\phi,1\dagger}, \alpha^{\phi,2\dagger}, \alpha^{\phi,3\dagger}\right)$ $|0,0,0)_\phi$ is identical to the action of the orbital part of the Ward operator on the orbital part of the vertex. The action of $W_m^{x,r}$ on the orbital part of the vertex may be computed with the help of the results obtained [14]. Thus, the anomaly of the quadratic part $W_m^{\phi,r}(2)$ is

$$
\kappa_m^{\phi,r}(2) = -\frac{1}{2} \sum_{k=1}^{m-1} k (m-k) G_{m-k\ k}^{rr} \tag{58}
$$

We have seen in appendix B, that G_{nm}^{rr} vanish for all $n + m = odd$ (this is a consequence of the cyclic property of the G coefficients); hence the above expression reduces to

$$
\kappa_m^{\phi,r}(2) = \frac{1 + (-1)^m}{2} \left[-\frac{1}{2} \sum_{k=1}^{m-1} k (m-k) G_{m-k\ k}^{rr} \right] \tag{59}
$$

The finite sum now may be evaluated by mathematical induction. For $m = 2$, we have

$$\kappa_2^{\phi,r}(2) = -\frac{1}{2}G_{11}^{rr} \tag{60}$$

From ref. [15], we have

$$G_{11}^{rr} = -\frac{a_1 b_1}{3} - \frac{1}{\pi}\sqrt{\frac{1}{3}}\left(a_1 \widetilde{E}_1^b - b_1 \widetilde{E}_1^a\right) \tag{61}$$

The explicit values of $\widetilde{E}_1^a$ and $\widetilde{E}_1^b$ are given in [15]. Thus using these values and the values of $a_1 = 2/3$ and $b_1 = 4/3$, we find

$$G_{11}^{rr} = -\frac{5}{3^3} \tag{62}$$

and so we find

$$\kappa_2^{\phi,r}(2) = -\frac{1}{2}\left[-\frac{5}{3^3}\right] = \frac{1}{2}\left(-\frac{5}{3^3}\right)\frac{2}{2}(-1)^{2/2} \tag{63}$$

For $m = 4$, equation (59) gives

$$\kappa_4^{\phi,r}(2) = -\frac{1}{2}\left[3G_{3\,1}^{rr} + 4G_{22}^{rr} + 3G_{1\,3}^{rr}\right] \tag{64}$$

From ref. [15], we have

$$G_{13}^{rr} = G_{31}^{rr} = \frac{2^5}{3^6} \tag{65}$$

and

$$G_{22}^{rr} = \frac{2a_2 b_2}{12} + \frac{1}{\pi}\sqrt{\frac{1}{3}}\left[a_2 \widetilde{S}_2^b - b_2 \widetilde{S}_2^a\right] \tag{66}$$

where

$$\widetilde{S}_2^a = \left[\widetilde{S}_1^a - \frac{2}{9}\sqrt{3}\pi\right]\frac{3}{2}a_2 - \frac{1}{3}\sqrt{3}\pi\left(-a_1 a_1 + \frac{1}{2}a_2\right) - \frac{1}{4}\sqrt{3}\pi\left(\frac{1}{2}a_1\right) \tag{67}$$

$$\widetilde{S}_2^b = \left[\widetilde{S}_1^b - \frac{4}{9}\sqrt{3}\pi\right]\frac{3}{4}b_2 - \frac{1}{3}\sqrt{3}\pi\left(-b_1 b_1 + \frac{1}{2}b_2\right) - \frac{3}{8}\sqrt{3}\pi\left(\frac{1}{2}b_1\right) \tag{68}$$

Using the explicit values of the a's, b's and $\widetilde{S}_1^a$ from [15], we find

$$\widetilde{S}_2^a = -\frac{1}{36}\sqrt{3}\pi\,(4\ln 2 - 4\ln 3 + 1) \tag{69}$$

and

$$\widetilde{S}_2^b = -\frac{1}{36}\sqrt{3}\pi\,(16\ln 2 - 16\ln 3 + 5) \tag{70}$$

and so equation (66) yields

$$G_{22}^{rr} = \frac{13}{2 \cdot 3^5} \tag{71}$$

Now substituting the values of G_{13}^{rr}, G_{31}^{rr} and G_{22}^{rr} into (64), we find

$$\kappa_4^{\phi,r}(2) = -\frac{1}{2}\left[\frac{10}{27}\right] = \frac{1}{2}\left(-\frac{5}{3^3}\right)\frac{4}{2}(-1)^{4/2} \tag{72}$$

Continuing this way we see that for $m = even$, the finite sum in (59) has the value

$$\sum_{k=1}^{m-1} k(m-k)\, G_{m-k\ k}^{rr} = -\left(-\frac{5}{3^3}\right)\frac{m}{2}(-1)^{m/2} \tag{73}$$

Thus we obtain

$$\kappa_m^{\phi,r}(2) = \frac{1+(-1)^m}{2}\left[\frac{1}{2}\left(-\frac{5}{3^3}\right)\frac{m}{2}(-1)^{m/2}\right] \tag{74}$$

The anomaly for the quadratic coordinate piece has been evaluated in [14]

$$\kappa_m^{x,r}(2) = \frac{1+(-1)^m}{2}\left[\frac{D}{2}\left(-\frac{5}{3^3}\right)\frac{m}{2}(-1)^{m/2}\right] \tag{75}$$

Combining equations (57), (74), and (75), we see that the total anomaly

$$\kappa_m^{x,r}(2) + \kappa_m^{\phi,r}(2) + \kappa_m^{\phi,r}(1) = 0 \tag{76}$$

vanishes in the critical dimension $D = 26$. This result provides a nontrivial consistency check on the validity of the comma theory.

Now we proceed to consider the Linear part of the Ward operator. Using the identity in (51), we can commute the Ward operator $W_m^{\phi,r}(1)$ through $V_\phi^{HS}\left(\alpha^{\phi,1\dagger}, \alpha^{\phi,2\dagger}, \alpha^{\phi,3\dagger}\right) >$, skipping some rather simple algebra, we find

$$\begin{aligned}
& W_m^{\phi,r}(1)\, V_\phi^{HS}\left(\alpha^{\phi,1\dagger}, \alpha^{\phi,2\dagger}, \alpha^{\phi,3\dagger}\right)|0>_{123}^{\phi} \\
= \quad & \left\{\sum_{k=1}^{\infty} \alpha_{-k}^{\phi,r} \cos\pi\frac{k+m}{2} + \sum_{k=1}^{m-1}\sum_{s=1}^{3}\sum_{q=0}^{\infty} k G_{kq}^{rs} a_{-q}^{\phi,s} \cos\pi\frac{m-k}{2}\right. \\
& + p_0^{\phi,r}\cos\frac{m\pi}{2} + \left(\frac{1}{2} - \frac{3}{2}m\right)\sum_{s=1}^{3}\sum_{q=0}^{\infty} m G_{mq}^{rs} a_{-q}^{\phi,s} \\
& + \sum_{s=1}^{3}\sum_{n=1}^{\infty} m\widetilde{N}_{mn}^{rs}\left[\sum_{k=1}^{\infty}\alpha_{-k-n}^{\phi,s}\cos\frac{k\pi}{2} + \left(\frac{1}{2} - \frac{3}{2}n\right)\alpha_{-n}^{\phi,s}\right] + \\
& \left.\sum_{s=1}^{3} m\widetilde{N}_{m0}^{rs}\left[\frac{1}{2}p_0^{\phi,s} + \sum_{k=1}^{\infty}\cos\frac{k\pi}{2}\alpha_{-k}^{\phi,s}\right]\right\} V_\phi^{HS}\left(\alpha^{\phi,1\dagger}, \alpha^{\phi,2\dagger}, \alpha^{\phi,3\dagger}\right)|0>_{123}^{\phi}
\end{aligned}$$

Making the identification $\alpha_0^{\phi,s} = p_0^{\phi,s}$ and then using the fact that

$$\sum_{q=1}^{m-1} q G_{q0}^{rs} \cos\pi\frac{m-q}{2} = \sum_{q=1}^{m-1} (m-q)\, G_{m-q\ 0}^{rs} \cos\pi\frac{q}{2} \tag{77}$$

the above expression becomes

$$\begin{aligned} = \quad & \left\{\sum_{s=1}^{3}\sum_{k=1}^{\infty}\left(\delta^{rs}\cos\pi\frac{k+m}{2}\right)\alpha_{-k}^{\phi,s} + \sum_{s=1}^{3}\sum_{k=1}^{\infty}\left(\sum_{q=1}^{m-1} qG_{qk}^{rs}\cos\pi\frac{m-q}{2}\right)a_{-k}^{\phi,s}\right. \\ & +\sum_{s=1}^{3}\sum_{k=1}^{\infty}\left(\frac{1}{2}-\frac{3}{2}m\right)mG_{mk}^{rs}a_{-k}^{\phi,s} + \sum_{s=1}^{3}\sum_{k=1}^{\infty}\left(m\widetilde{N}_{m0}^{rs}\cos\frac{k\pi}{2}\right)\alpha_{-k}^{\phi,s} \\ & +\sum_{s=1}^{3}\sum_{n=1}^{\infty} m\widetilde{N}_{mn}^{rs}\sum_{k=1}^{\infty}\alpha_{-k-n}^{\phi,s}\cos\frac{k\pi}{2} + \sum_{s=1}^{3}\sum_{k=1}^{\infty} m\widetilde{N}_{mk}^{rs}\left(\frac{1}{2}-\frac{3}{2}k\right)\alpha_{-k}^{\phi,s} \\ & +\sum_{s=1}^{3}\delta^{rs}\cos\frac{m\pi}{2} + \frac{1}{2}m\widetilde{N}_{m0}^{rs} + \left(\frac{1}{2}-\frac{3}{2}m\right)mG_{m0}^{rs} + \\ & \left.\left.\left.\sum_{q=1}^{m-1}(m-q)\,G_{m-q\ 0}^{rs}\cos\pi\frac{q}{2}\right]\right]\right\} p_0^{\phi,s} V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0)_{123}^{\phi} \end{aligned}$$

If we now let $k+n \longrightarrow q$ in the double sum $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(....)$, so that

$$\begin{aligned} & \sum_{n=1}^{\infty} m\widetilde{N}_{mn}^{rs}\sum_{k=1}^{\infty}\alpha_{-k-n}^{\phi,s}\cos\frac{k\pi}{2} \\ = \quad & \sum_{n=1}^{\infty}\sum_{k=1+n}^{\infty} m\widetilde{N}_{mn}^{rs}\alpha_{-k}^{\phi,s}\cos\pi\frac{k-n}{2} = \sum_{k=1}^{\infty}\sum_{n=1}^{k-1} m\widetilde{N}_{mn}^{rs}\alpha_{-k}^{\phi,s}\cos\pi\frac{k-n}{2} \end{aligned}$$

(to see the last equality you only need to expand both sides and compare terms), we obtain

$$\begin{aligned} & W_m^{\phi,r}(1)\, V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0,0,0>_\phi \\ = \quad & \left[\sum_{s=1}^{3}\sum_{k=1}^{\infty}\Omega_{mk}^{rs}\alpha_{-k}^{\phi,s} + \sum_{s=1}^{3}\Omega_{m0}^{rs}\ p_0^{\phi,s}\right] V_\phi^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0,0,0>_\phi \end{aligned} \tag{78}$$

where for $k, m = 1, 2, 3, ...$

$$\begin{aligned} \Omega_{mk}^{rs} \quad \equiv \quad & \delta^{rs}\cos\pi\frac{k+m}{2} + \sum_{n=1}^{m-1} nG_{nk}^{rs}\cos\pi\frac{m-n}{2} + \left(\frac{1}{2}-\frac{3}{2}m\right)mG_{mk}^{rs} \\ & +m\widetilde{N}_{m0}^{rs}\cos\frac{k\pi}{2} + \sum_{n=1}^{k-1} m\widetilde{N}_{mn}^{rs}\cos\pi\frac{k-n}{2} + m\widetilde{N}_{mk}^{rs}\left(\frac{1}{2}+\frac{3}{2}k\right) \end{aligned} \tag{79}$$

and for $k=0, m=1,2,3,...$

$$\Omega^{rs}_{m0} \equiv \delta^{rs}\cos\frac{m\pi}{2}+\frac{1}{2}m\widetilde{N}^{rs}_{m0}+\left(\frac{1}{2}-\frac{3}{2}m\right)mG^{rs}_{m0}+\sum_{n=1}^{m-1}(m-n)\,G^{rs}_{m-n\ 0}\cos\pi\frac{n}{2} \tag{80}$$

In obtaining equation (78) we have labeled the dummy index q as n. Thus to establish that $W^{\phi,r}_m(1)\,V^{HS}_\phi\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0>^\phi_{123}=0$, we need to prove that $\sum_{s=1}^3\sum_{k=1}^\infty\Omega^{rs}_{mk}\alpha^{\phi,s}_{-k}\ V^{HS}_\phi\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0\ >^\phi_{123}=\ 0$ and $\sum_{s=1}^3\Omega^{rs}_{m0}p^{\phi,s}_0$ $V^{HS}_\phi\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0\ >^\phi_{123}=\ 0$. We observe that for $k\neq 0$, the states $\alpha^{\phi,s}_{-k}V^{HS}_\phi\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)|0>^\phi_{123}$ are all linearly independent for $s=1,2,3$ and $k=1,2,...$, thus the only way for this part to vanish is for Ω^{rs}_{mk} to be identically zero for all values of $s=1,2,3$ and $k=1,2,...$ That is we need to show that

$$\Omega^{rs}_{mk}=0 \tag{81}$$

is true for all values of $r,s=1,2,3$ and $k,m=1,2,...$ For the second part we need to prove that

$$\sum_{s=1}^{3}\Omega^{rs}_{m0}p^{\phi,s}_0=0 \tag{82}$$

for $m=1,2,...$ Using the conservation of momentum (or more precisely the ghost number conservation) to eliminate [3] $p^{\phi,3}_0$, we see at once that to establish the above equation we only need to show that the following identities

$$\begin{aligned}\left(\Omega^{r1}_{m0}-\Omega^{r3}_{m0}\right) &= 0 \qquad (83)\\ \left(\Omega^{r2}_{m0}-\Omega^{r3}_{m0}\right) &= 0 \qquad (84)\end{aligned}$$

are satisfied for $r=1,2,3$, $m=1,2,...$ To prove that Ω^{rs}_{mk} in equation (79) vanish for all values of r,s,m,k is quite cumbersome. Unfortunately, it appears to be no short cuts in proving $\Omega^{rs}_{mk}=0$ and so we are forced to prove that $\Omega^{rs}_{mk}=0$ by brute force. The identities in (83) and (84) may be proven with the help of the properties of the coefficients G^{rs}_{nm} and $\widetilde{N}^{rs}_{nm}$ established in [15]. First let us concentrate on the identities in (83) and (84). For $m=odd=2k+1>0$, equation (80) reduces to

$$\begin{aligned}\Omega^{rs}_{2k+10} &= \frac{1}{2}(2k+1)\,\widetilde{N}^{rs}_{2k+10}+\left(\frac{1}{2}-\frac{3}{2}(2k+1)\right)(2k+1)\,G^{rs}_{2k+10}\\ &\quad+\sum_{n=1}^{2k}(2k+1-n)\,G^{rs}_{2k+1-n\ 0}\cos\pi\frac{n}{2} \qquad (85)\end{aligned}$$

For $r=s$, we have $G^{rr}_{odd\ even}=\widetilde{N}^{rr}_{odd\ even}=0$ (see [15]), and so the above expression vanish since for $n=even$, $G^{rs}_{2k+1-n\ 0}=0$ and for $n=odd$, $\cos\pi n/2$ vanish. Thus for $r=s$, we have

$$\Omega^{rr}_{2k+10}=0$$

[3] If we choose to eliminate $p^{\phi,1}_0$ or $p^{\phi,2}_0$ instead of $p^{\phi,3}_0$ the conclusions remain the same.

It follows from the above identity and (83) and (84) that to prove the identities in (83) and (84) for $m = odd = 2k+1 > 0$, we have to establish that

$$\Omega^{12}_{2k+1,0} = \Omega^{23}_{2k+1,0} = \Omega^{31}_{2k+1,0} = \Omega^{13}_{2k+1,0} = \Omega^{32}_{2k+1,0} = \Omega^{21}_{2k+1,0} = 0 \tag{86}$$

It is not hard to check that the above identities are satisfied for $2k+1 = 1, 2, 3, ..,$ by explicit substitution. The proof for a general value of $2k+1$, can be established by mathematical induction. We have proved that in fact these identities are satisfied for all values $2k+1$; however to present the proof here for all cases will occupy too much space. To illustrate the proof we will present here the complete proof for one of them. Let us consider

$$\Omega^{13}_{2k+10} = 0 \tag{87}$$

From (85) we have

$$\begin{aligned} \Omega^{13}_{2k+10} &= \frac{1}{2}(2k+1)\widetilde{N}^{13}_{2k+10} + \left(\frac{1}{2} - \frac{3}{2}(2k+1)\right)(2k+1)\, G^{13}_{2k+10} \\ &\quad + \sum_{n=1}^{2k} (2k+1-n)\, G^{13}_{2k+1-n\ 0} \cos \pi \frac{n}{2} \end{aligned} \tag{88}$$

From [15], we have

$$G^{13}_{2k+10} = -\frac{1}{\sqrt{3}}(-)^k \frac{a_{2k+1}}{2k+1} \tag{89}$$

$$\widetilde{N}^{13}_{2k+10} = -\frac{1}{\sqrt{3}}(-)^k \frac{b_{2k+1}}{2k+1} \tag{90}$$

And so the above equation becomes

$$\Omega^{13}_{2k+10} = \frac{1}{\sqrt{3}}(-)^k \left[-\frac{1}{2}b_{2k+1} + (3k+1)\, a_{2k+1} - \sum_{m=1}^{k} a_{2k+1-2m}\right] \tag{91}$$

To proceed further, we need to eliminate either b or a from the above expression. This can be established with the help of the mixed recursion relation

$$\frac{2}{3}b_n = (-1)^n \left[(n+1)\, a_{n+1} - 2na_n + (n-1)\, a_{n-1}\right] \tag{92}$$

which can be established with the help of contour integration (see [15] for similar procedures). Thus setting $n = 2k+1$ in the recursion relation and substituting for b_{2k+1}, the above expression in (91) reduces to

$$\Omega^{13}_{2k+10} = \frac{1}{\sqrt{3}}(-)^k \left[\frac{3}{2}(k+1)\, a_{2k+2} + \frac{3}{2}ka_{2k} - \frac{1}{2}a_{2k+1} - \sum_{m=1}^{k} a_{2k+1-2m}\right] \tag{93}$$

The expression inside the square bracket can be shown to vanish for all values of $k = 0, 1, 2, 3, ...$ with the help of the recursion relation in [15]

$$\frac{2}{3}a_n = (n+1)\, a_{n+1} - (n-1)\, a_{n-1} \tag{94}$$

To see this we set $n = 2k+1$ in the above recursion relation and then substituting in (93) to obtain

$$\Omega^{13}_{2k+10} = \frac{1}{\sqrt{3}} (-)^k \left[3ka_{2k} - \sum_{m=1}^{k} a_{2k+1-2m} \right] \tag{95}$$

Setting $n = 2k - 1$ in (94) and then substituting in (95), we find

$$\Omega^{13}_{2k+10} = \frac{1}{\sqrt{3}} (-)^k \left[3\,(k-1)\, a_{2k-2} - \sum_{m=2}^{k} a_{2k+1-2m} \right] \tag{96}$$

At this point it is not hard to show by explicit substitution that both expressions inside the square brackets in (95) and (96) vanish for $k = 0, 1, 2$. Now we assume that both expressions inside the square brackets in (95) and (96) vanish for k. We let $k \to k+1$ in (96) so that the expression inside the square bracket of (96) becomes

$$3ka_{2k} - \sum_{m=2}^{k+1} a_{2k+1+2(1-m)} \tag{97}$$

Letting $1 - m = n$, and then letting $n \to -n$, the above expression becomes

$$3ka_{2k} - \sum_{m=2}^{k+1} a_{2k+1+2(1-m)} = 3ka_{2k} - \sum_{n=1}^{k} a_{2k+1-2n} = 0 \tag{98}$$

where the second equality follows from the fact that the expression inside the square bracket in (95) vanishes for all values of k. Thus the expression inside the square bracket in (96) vanishes identically by mathematical induction and the desired result follows at once. In fact using this procedure we have checked that all the identities in (86) are satisfied.

For the case of $m = even$, one can show using similar procedure to the one used to establish (86) and the following properties of the G and $\widetilde{N}$ coefficients

$$G^{13}_{2k\ 0} = G^{12}_{2k\ 0}, \quad G^{13}_{2k+1\ 0} = -G^{12}_{2k+1\ 0} \tag{99}$$

$$\widetilde{N}^{13}_{2k\ 0} = \widetilde{N}^{12}_{2k\ 0}, \quad \widetilde{N}^{13}_{2k+1\ 0} = -\widetilde{N}^{12}_{2k+1\ 0} \tag{100}$$

that the only none trivial identity in equations (83) and (84) is

$$\Omega^{11}_{2k\ 0} - \Omega^{13}_{2k\ 0} = 0 \tag{101}$$

Using equation (80), the left hand side of the above equation becomes

$$\begin{aligned} \Omega^{11}_{2k\ 0} - \Omega^{13}_{2k\ 0} &= (-1)^k + k \left(\widetilde{N}^{11}_{2k\ 0} - \widetilde{N}^{13}_{2k\ 0} \right) + (1-6k)\, k \left(G^{11}_{2k\ 0} - G^{13}_{2k\ 0} \right) \\ &\quad + \sum_{n=1}^{k-1} (2k-2n)\, (-1)^n \left(G^{11}_{2k-2n\ 0} - G^{13}_{2k-2n\ 0} \right) \end{aligned} \tag{102}$$

Using the values of the G and $\widetilde{N}$ coefficients computed in [15]

$$\begin{aligned} G^{11}_{2k\ 0} &= \frac{2}{3}(-)^k \frac{a_{2k}}{2k}, \qquad G^{13}_{2k\ 0} = -\frac{1}{3}(-)^k \frac{a_{2k}}{2k} \\ G^{11}_{2k-2n\ 0} &= \frac{2}{3}(-)^{k-n} \frac{a_{2k-2n}}{2k-2n}, \qquad G^{13}_{2k-2n\ 0} = -\frac{1}{3}(-)^{k-n} \frac{a_{2k-2n}}{2k-2n} \\ \widetilde{N}^{11}_{2k\ 0} &= -\frac{2}{3}(-)^k \frac{b_{2k}}{2k}, \qquad \widetilde{N}^{13}_{2k\ 0} = \frac{1}{3}(-)^k \frac{b_{2k}}{2k} \end{aligned}$$

the above expression becomes

$$\Omega^{11}_{2k\ 0} - \Omega^{13}_{2k\ 0} = (-1)^k \left[1 - \frac{1}{2}b_{2k} + \frac{1}{2}(1-6k)\, a_{2k} + \sum_{n=1}^{k-1} a_{2k-2n}\right] \tag{103}$$

Now to prove (101), we have to show that the expression inside the square bracket in the above expression vanish for all k. The proof is very similar to the proof of the identities in (86). We can use the mixed recursion relation in (92) to eliminate b_{2k} from the expression inside the square bracket on the right hand side of the above equation, then with the help of the recursion relation in (94) one can show, by mathematical induction, that

$$\left[1 - \frac{1}{2}b_{2k} + \frac{1}{2}(1-6k)\, a_{2k} + \sum_{n=1}^{k-1} a_{2k-2n}\right] = 0 \tag{104}$$

for all values of k. Consequently the identity in (101) follows at once.

We still need to prove the identity in (81). Unfortunately this identity is quite difficult to prove; the difficulty arises because of the finite sums over the G and $\widetilde{N}$ coefficients. There is no obvious way of proving this identity in a clean way other than by brute force. However let us start by working out few specific cases. We first consider Ω^{rs}_{11}. letting $m = k = 1$ in (79) we have

$$\Omega^{rs}_{11} = -\delta^{rs} - G^{rs}_{11} + 2\widetilde{N}^{rs}_{11} \tag{105}$$

For $r = s$; we have $G^{rr}_{11} = -5/3^3$ and $\widetilde{N}^{rs}_{11} = 11/3^3$ (see [15]) so the right hand side of the above equation vanish for $r = s$. Thus $\Omega^{rr}_{11} = 0$ for $r = 1, 2, 3$. For $r = 1,\ s = 2, 3$, we have $G^{12}_{11} = G^{13}_{11} = 2^4/3^3$, $\widetilde{N}^{12}_{11} = \widetilde{N}^{13}_{11} = 2^3/3^3$ (see [15]). Using these values we see that $-\delta^{rs} - G^{rs}_{11} + 2\widetilde{N}^{rs}_{11} = 0$ for $r = 1,\ s = 2, 3$ and so we have $\Omega^{12}_{11} = \Omega^{13}_{11} = 0$. The cyclic symmetries of G^{rs}_{nm} and $\widetilde{N}^{rs}_{nm}$ now imply that the right hand side of (105) also vanish for $(r, s) = (2, 3)$, $(3, 1)$, $(3, 2)$ and $(2, 1)$. Thus we have established that $\Omega^{rs}_{11} = 0$ for all r and s. We will work out few more case; however in what follows we shall not refer to the exact equations explicitly for the G^{rs}_{nm} and $\widetilde{N}^{rs}_{nm}$ coefficients since they are are derived in [15] and it will not be much of a task for the reader to look them up. Let us next consider the case when $k + m = odd$. For $k = odd$ and $m = even$ equation (79) yields

$$
\begin{aligned}
\Omega^{rs}_{2m\ 2k+1} &= \sum_{n=1}^{m-1} 2nG^{rs}_{2n\ 2k+1}(-1)^{m-n} + (1-6m)\, mG^{rs}_{2m\ 2k+1} \\
&\quad + \sum_{n=1}^{k} 2m\widetilde{N}^{rs}_{2m\ 2n-1}(-1)^{k-n+1} + m\widetilde{N}^{rs}_{2m\ 2k+1}(1+3(2k+1)) \qquad (106)
\end{aligned}
$$

First consider $r = s$; in this case $G^{rr}_{n+m=ood} = \widetilde{N}^{rr}_{n+m=odd} = 0$ and the right hand side vanish; hence $\Omega^{rr}_{2m\ 2k+1} = 0$. Likewise one can show that $\Omega^{rr}_{2m+1\ 2k} = 0$. Next we consider $\Omega^{12}_{2m\ 2k+1} = \Omega^{23}_{2m\ 2k+1} = \Omega^{31}_{2m\ 2k+1}$, where the equality here follows from the cyclic symmetry of the G^{rs}_{nm} and $\widetilde{N}^{rs}_{nm}$ coefficients. Thus it suffice in this case to show that $\Omega^{12}_{2m\ 2k+1} = 0$. To prove this put $r = 1$, $s = 2$ in equation (106)

$$
\begin{aligned}
\Omega^{12}_{2m\ 2k+1} &= (-1)^m \sum_{n=1}^{m-1} 2nG^{12}_{2n\ 2k+1}(-1)^n + (1-6m)\, mG^{12}_{2m\ 2k+1} + (-1)^k \\
&\quad \sum_{n=0}^{k-1} 2m\widetilde{N}^{12}_{2m\ 2n+1}(-1)^n + m\widetilde{N}^{12}_{2m\ 2k+1}(1+3(2k+1)) \qquad (107)
\end{aligned}
$$

At this point we can use the explicit values of the $G^{12}_{2n\ 2k+1}$, $G^{12}_{2m\ 2k+1}$, $\widetilde{N}^{12}_{2m\ 2n-1}$ and $\widetilde{N}^{12}_{2m\ 2k+1}$coefficients computed in [15] and carry the calculation to the end however this will take too much space. So at this stage let us work out specific cases. For $m = 1$, $k = 0$, the above expression becomes

$$
\Omega^{12}_{2\ 1} = -5G^{12}_{2\ 1} + 4\widetilde{N}^{12}_{2\ 1}
$$

From [15], we have $G^{12}_{2\ 1} = 2^5/3^4\sqrt{3}$, $\widetilde{N}^{12}_{2\ 1} = -2^3 \cdot 5/3^4\sqrt{3}$ and so the right hand side of the above equation is identically zero. Hence, we have established that $\Omega^{12}_{2\ 1} = 0$. For $m = 1$, $k = 1$, equation (107) gives

$$
\Omega^{12}_{2\ 3} = -5G^{12}_{2\ 3} - 2\widetilde{N}^{12}_{2\ 1} + 10\widetilde{N}^{12}_{2\ 3} \qquad (108)
$$

From [15], we have $G^{12}_{2\ 3} = -4 \cdot 2^3 \cdot 5/3^6\sqrt{3}$, $\widetilde{N}^{12}_{2\ 1} = -2^3 \cdot 5/3^4\sqrt{3}$, $\widetilde{N}^{12}_{2\ 3} = -19 \cdot 2^3/3^6\sqrt{3}$. Substituting these values in the above expression yields $\Omega^{12}_{2\ 3} = 0$.

From these examples, we see that the difficulty involved in constructing a general proof of these identities. However, we still can describe the steps involved in the proof and give some of the details. Using the G^{rs}_{nm} and $\widetilde{N}^{rs}_{nm}$ coefficients computed in [15], one can show that the only nontrivial identities in (81) are the following four

$$
\Omega^{12}_{2m\ 2k+1} = \Omega^{12}_{2m+1\ 2k} = \Omega^{11}_{2m\ 2k} = \Omega^{11}_{2m+1\ 2k+1} \qquad (109)
$$

and all the other identities are either trivially satisfied or can be deduced from these four identities. To illustrate the proof of the above identities we consider

$\Omega^{12}_{2m\ 2k+1}$. Substituting the explicit values of (see [15])

$$G^{12}_{2n2m+1} = \frac{(-)^{n+m}}{2\sqrt{3}}\left[\frac{a_{2n}b_{2m+1}-b_{2n}a_{2m+1}}{(2n)+(2m+1)}+\frac{a_{2n}b_{2m+1}+b_{2n}a_{2m+1}}{(2n)-(2m+1)}\right] \quad (110)$$

$$\widetilde{N}^{12}_{2n2m+1} = \frac{(-1)^{n+m}}{2\sqrt{3}}\left[\frac{b_{2n}a_{2m+1}-a_{2n}b_{2m+1}}{2n+(2m+1)}+\frac{b_{2n}a_{2m+1}+a_{2n}b_{2m+1}}{2n-(2m+1)}\right] \quad (111)$$

into equation (107) , and skipping some algebra we obtain

$$\begin{aligned} &\frac{2\sqrt{3}}{(-)^{m+k}}\Omega^{12}_{2m\ 2k+1} \\ =\ & \sum_{n=0}^{m-1} 2n\left[\frac{a_{2n}b_{2k+1}-b_{2n}a_{2k+1}}{(2n)+(2k+1)}+\frac{a_{2n}b_{2k+1}+b_{2n}a_{2k+1}}{(2n)-(2k+1)}\right] \\ +\ & \sum_{n=0}^{k-1} 2m\left[\frac{b_{2m}a_{2n+1}-a_{2m}b_{2n+1}}{2m+(2n+1)}+\frac{b_{2m}a_{2n+1}+a_{2m}b_{2n+1}}{2m-(2n+1)}\right] \\ & -2m\frac{4a_{2m}b_{2k+1}+b_{2m}a_{2k+1}+6ka_{2m}b_{2k+1}-6ma_{2m}b_{2k+1}}{2k-2m+1} \end{aligned}$$

At this stage we have checked by explicit substitution of the a's and the b's that the right hand side vanish for the first few low values of m and k. Theses consistency checks are important to ensure that we are on the right track. Now we use the mixed recursion relation in (92) to eliminate the b' in favor of the a's; hence using (92), the expression becomes

$$\begin{aligned} &\frac{2\sqrt{3}}{(-)^{m+k}}\Omega^{12}_{2m\ 2k+1} \qquad (112) \\ =\ & \sum_{n=0}^{m-1}\frac{-1}{4k^2+4k-4n^2+1}\{6na_{2k+1}a_{2n+1}-6na_{2k+1}a_{2n-1}-24n^2a_{2n}a_{2k+2} \\ & +12n^2a_{2k+1}a_{2n-1}+12n^2a_{2k+1}a_{2n+1}-24kn^2a_{2k}a_{2n}-12kna_{2k+1}a_{2n-1} \\ & +12kna_{2k+1}a_{2n+1}-24kn^2a_{2n}a_{2k+2}+24kn^2a_{2k+1}a_{2n-1}+24kn^2a_{2k+1}a_{2n+1}\} \\ & \sum_{n=0}^{k-1}\frac{1}{-4m^2+4n^2+4n+1}\{12ma_{2m}a_{2n+2}-12ma_{2m}a_{2n+1}+48m^3a_{2m}a_{2n+1} \\ & +12m^2a_{2m-1}a_{2n+1}-12m^2a_{2m+1}a_{2n+1}-24m^3a_{2m-1}a_{2n+1}-24m^3a_{2m+1}a_{2n+1} \\ & -48mna_{2m}a_{2n+1}+36mna_{2m}a_{2n+2}+24mn^2a_{2m}a_{2n}-48mn^2a_{2m}a_{2n+1} \\ & +24mn^2a_{2m}a_{2n+2}+12mna_{2m}a_{2n}\} \\ & +2\frac{m}{2k-2m+1}\{12a_{2m}a_{2k+2}-12a_{2m}a_{2k+1}-3a_{2k+1}a_{2n+1}+3a_{2k+1}a_{2n+2} \\ & -42ka_{2m}a_{2k+1}+30ka_{2m}a_{2k+2}+18ma_{2m}a_{2k+1}-18ma_{2m}a_{2k+2}+3na_{2n}a_{2k+1} \\ & +18k^2a_{2k}a_{2m}-6na_{2k+1}a_{2n+1}+3na_{2k+1}a_{2n+2}-36k^2a_{2m}a_{2k+1}+18k^2 \\ & a_{2m}a_{2k+2}+12ka_{2k}a_{2m}+36kma_{2m}a_{2k+1}-18kma_{2m}a_{2k+2}-18kma_{2k}a_{2m}\} \end{aligned}$$

At this point we have checked by explicit substitution that the right hand side is zero for the first few values of m and k. So we will prove this identity by mathematical induction. Let us assume the right hand side of the above equation vanish for given m and k, then for $m \to m+1$ and $k \to k+1$, the right hand side (RHS) reads

$$
\begin{aligned}
\text{RHS} \;=\; & \sum_{n=0}^{m} \frac{-6n}{4k^2+12k-4n^2+9}\{3a_{2k+3}a_{2n+1}-3a_{2k+3}a_{2n-1}-8na_{2n}a_{2k+4} \\
& -2ka_{2k+3}a_{2n-1}+2ka_{2k+3}a_{2n+1}+6na_{2k+3}a_{2n-1}+6na_{2k+3}a_{2n+1} \\
& -4kna_{2n}a_{2k+2}-4kna_{2n}a_{2k+4}+4kna_{2k+3}a_{2n-1}+4kna_{2k+3}a_{2n+1}\} \\
& +\sum_{n=0}^{k} \frac{1}{-4\left(m+1\right)^2+4n^2+4n+1}\{ \\
& 12a_{2m+2}\left(m+1\right)\left(4a_{2n+1}m^2+8a_{2n+1}m+3a_{2n+1}+a_{2n+2}\right) \\
& -12a_{2n+1}\left(m+1\right)^2\left(2ma_{2m+1}+2ma_{2m+3}+a_{2m+1}+3a_{2m+3}\right) \\
& -12na_{2m+2}\left(m+1\right)\left(4na_{2n+1}-2na_{2n}+4a_{2n+1}-3a_{2n+2}\right) \\
& +12n\left(2na_{2n+2}+a_{2n}\right)a_{2m+2}\left(m+1\right)\} \\
& +\frac{2\left(m+1\right)}{2\left(k+1\right)-2\left(m+1\right)+1}\{12a_{2k+4}a_{2m+2}-3a_{2k+3}a_{2n+1} \\
& -12a_{2k+3}a_{2m+2}+3a_{2k+3}a_{2n+2}+3na_{2n}a_{2k+3}-18a_{2k+4}a_{2m+2} \\
& -24a_{2k+3}a_{2m+2}-42ka_{2k+3}a_{2m+2}+30ka_{2k+4}a_{2m+2}+18ma_{2k+3}a_{2m+2} \\
& -18ma_{2k+4}a_{2m+2}+18a_{2k+2}a_{2m+2}-36a_{2k+3}a_{2m+2}+36ka_{2k+2}a_{2m+2} \\
& -72ka_{2k+3}a_{2m+2}-6na_{2k+3}a_{2n+1}+3na_{2k+3}a_{2n+2}+18k^2a_{2k+2}a_{2m+2} \\
& -36k^2a_{2k+3}a_{2m+2}+6a_{2m+2}\left(k+1\right) \\
& \left(3ka_{2k+4}-3ma_{2k+2}+6ma_{2k+3}-3ma_{2k+4}-a_{2k+2}+6a_{2k+3}\right)\}
\end{aligned}
$$

This expression reduces to the right hand side of (112). To see this one needs only to use the recursion relations in (94), to reduces the indices to those appearing in (112). We have checked that this is indeed true, however the calculation is quite messy to include here, otherwise straight forward. Thus the right hand side of (112) vanish identically by induction for all values of m and k. Likewise one can prove the remaining identities; the procedure is straight forward but it takes countless number of pages and so we shall not include any more details here.

The full Ward-like identity now reads

$$
\left[L_m^{x+\phi,r}+\sum_{s=1}^{3}\sum_{n=0}^{\infty} m\widetilde{N}_{mn}^{rs}L_{-n}^{x+\phi,s}\right]|V_{HS}^{x+\phi}>=0, \quad m=1,2,... \tag{113}
$$

We have seen that the coordinate and the ghost anomaly cancel; thus the above equation contain no anomaly term.

3 The proof of the K invariance, the $BRST$ invariance and Bose-Fermi Equivalence

The K_m invariance of the comma interaction three vertex follows at once from the Ward-like identity. Summing over the string index r, we have

$$\sum_{r=1}^{3}\left[L_m^{x+\phi,r}+\sum_{s=1}^{3}\sum_{n=0}^{\infty}m\widetilde{N}_{mn}^{rs}L_{-n}^{x+\phi,s}\right]|V_{HS}^{x+\phi}>=0, \quad m=1,2,... \tag{114}$$

Using the identity $\sum_{r=1}^{3}m\widetilde{N}_{mn}^{rs}=(-1)^{m+1}\,\delta_{mn}$, which can be established by contour integration, and then renaming the dummy index s as r, the above equation reduces to

$$\sum_{r=1}^{3}\left[L_m^{x+\phi,r}-(-1)^m\,L_{-m}^{x+\phi,r}\right]|V_{HS}^{x+\phi}>=0, \quad m=1,2,... \tag{115}$$

The expression inside the square bracket in the above equation is by definition the Virasoro generator K_m. Thus $|V_{HS}^{x+\phi}>$ is invariant under the subgroup of conformal transformations, generated by the Virasoro generators

$$K_m=\sum_{r=1}^{3}\left[L_m^{x+\phi,r}-(-1)^m\,L_{-m}^{x+\phi,r}\right] \tag{116}$$

To complete the proof of equivalence, we still need to show the $BRST$ (Q) invariance of the comma three vertex. Unfortunately a direct proof that follows from equation (113) is quite cumbersome due to the presence of the $\frac{1}{2}L_m^{\phi,r}$ term in the definition of the $BRST$ charge. Thus we need to evaluate the action of the $BRST$ on the comma three vertex directly. Recall that the total three string $BRST$ charge is the sum of the $BRST$ charges corresponding to the individual strings; that is,

$$Q=\sum_{r=1}^{3}Q^r \tag{117}$$

where

$$\begin{aligned}Q^r &= \sum_{m=1}^{\infty}\left[c_{-m}^r\left(L_m^{x,r}+\frac{1}{2}L_m^{\phi,r}\right)+\left(L_{-m}^{x,r}+\frac{1}{2}L_{-m}^{\phi,r}\right)c_m^r\right]\\ &\quad+\left(L_0^{x,r}+\frac{1}{2}L_0^{\phi,r}-1\right)c_0^r\end{aligned} \tag{118}$$

To evaluate the action of $BRST$ charge on the full comma three vertex we use the c-overlaps satisfied by the comma ghost vertex [17, 6]

$$\left[c_m^r-\sum_{s=1}^{3}\sum_{n=1}^{\infty}\widetilde{N}_{mn}^{rs}c_{-n}^s\right]|V_{HS}^{\phi}>=0 \tag{119}$$

With the help of (119), the proof of the $BRST$ invariance follows along the same lines of references [17, 6]. Thus when acting with the $BRST$ charge on the full comma vertex, the operator parts cancel, leaving

$$\begin{aligned} Q|V_{HS}^{x+\phi} > &= \sum_{r=1}^{3}\sum_{m=1}^{\infty} c_{-m}^{r}\left[\left(\kappa_m^{x,r}+\frac{1}{2}\kappa_m^{\phi,r}\right)-\left(\frac{1}{2}m^2\widetilde{N}_{0m}^{rr}\right.\right. \\ &\left.\left.+\frac{1}{2}\sum_{k=1}^{m-1}(m+k)\,(m-k)\,\widetilde{N}_{m\;m-k}^{rr}\right)\right]|V_{HS}^{x+\phi}> \end{aligned} \tag{120}$$

The expression inside the second parenthesis can be computed easily and it is found to be the anomaly of the fermionic ghost [17, 6]. Thus the coefficient of c_{-m}^{r} is

$$\kappa_m^{x,r}+\frac{1}{2}\kappa_m^{\phi,r}+\frac{1}{2}\kappa_m^{c,r}=0 \tag{121}$$

where in obtaining the above result we used the fact that the coordinate anomaly is the negative of the ghost anomaly regardless of the ghost representation. This result is the final step in the proof of equivalence.

A Sums encountered in the main body of the paper

Here we give the definitions of the sums encountered in the proof of the Ward-like identities, the K invariance, the $BRST$ invariance and their values without showing the algebra leading to these explicit values. For the computation of various sums appearing here the reader should consult references [17, 18, 15].

A.1 Sums of the first type

We define the quantities $O_{n=2k}^{u(q,p)}$ and $E_{n=2k+1}^{u(q,p)}$ by

$$O_{n=2k}^{u(q,p)} = \sum_{m=2l+1=1}^{\infty}\frac{u_m^{q/p}}{n+m}\ ,\ n\geq 0 \tag{122}$$

$$E_{n=2k+1}^{u(q,p)} = \sum_{m=2l=0}^{\infty}\frac{u_m^{q/p}}{n+m}\ ,\ n>0 \tag{123}$$

where $u_m^{q/p}$ are the Taylor modes appearing in the expansion

$$\left(\frac{1+z}{1-z}\right)^{q/p} = \sum_{m=0}^{\infty}u_n^{q/p}z^n \tag{124}$$

Serious exercise in contour integration leads to

$$O^{u(q,p)}_{n=2k>0} = \sum_{m=2l+1=1}^{\infty} \frac{u^{q/p}_m}{n+m} = \frac{\pi}{2} \frac{1}{\sin\left(\pi\frac{q}{p}\right)} u^{q/p}_{n=2k}, \tag{125}$$

$$O^{u(q,p)}_0 = \frac{\pi}{2} \tan\left(\pi\frac{q}{2p}\right) \tag{126}$$

Likewise for $n = odd = 2k+1 > 0$, we find

$$E^{u(q,p)}_{n=2k+1} = \sum_{m=2l=0}^{\infty} \frac{u^{q/p}_m}{n+m} = \frac{\pi}{2} \frac{1}{\sin\left(\pi\frac{q}{p}\right)} u^{q/p}_{n=2k+1} \tag{127}$$

respectively, which are the desired results. For negative values of n, one obtains

$$E^{u(q,p)}_{-n=-(2k+1)} = -\cos\left(\frac{\pi q}{p}\right) E^{u(q,p)}_{n=2k+1}, \quad n > 0 \tag{128}$$

and

$$O^{u(q,p)}_{-n=-2k} = -\cos\left(\frac{\pi q}{p}\right) O^{u(q,p)}_{n=2k}, \quad n > 0 \tag{129}$$

respectively. So far we have evaluated the sums defined in (122) and (123) under the restriction $n + m = odd$; now we would like to relax this restriction, which brings us to the sums of the second type.

A.2 Sums of the second type

We define the quantities $S^{u(q,p)}_n$ (sums of the second type) by

$$S^{u(q,p)}_{n=2k+1} \equiv O^{u(q,p)}_{n=2k+1} = \sum_{m=2l+1=1}^{\infty} \frac{u^{q/p}_m}{n+m} \tag{130}$$

$$S^{u(q,p)}_{n=2k} \equiv E^{u(q,p)}_{n=2k} = \sum_{m=2l=0}^{\infty} \frac{u^{q/p}_m}{n+m} \tag{131}$$

An exercise in contour integration, generating functions, differential equations and difference equations, leads to

$$\begin{aligned} S^{u(q/p)}_n &= \frac{p}{2q}\left[\frac{q}{p}\left(\beta\left(1-\frac{q}{p}\right)+\beta\left(1+\frac{q}{p}\right)\right)\right] u^{q/p}_n \\ &+ \frac{p}{2q} \sum_{m=0}^{n-1} (-)^m \frac{u^{q/p}_m u^{q/p}_{n-m-1}}{m+1} \end{aligned} \tag{132}$$

A.3 Sums of the third type

We define the quantities $\tilde{O}{}^{u(q,p)}_{n=2k}$ and $\tilde{E}{}^{u(q,p)}_{n=2k+1}$ (sums of the third type) by

$$\tilde{O}{}^{u(q,p)}_{n=2k} = \sum_{m=2l+1=1}^{\infty} \frac{u_m^{q/p}}{(n+m)^2} \tag{133}$$

$$\tilde{E}{}^{u(q,p)}_{n=2k+1} = \sum_{m=2l=0}^{\infty} \frac{u_m^{q/p}}{(n+m)^2} \tag{134}$$

Explicit computation gives

$$\begin{aligned} \tilde{E}{}^{u(p,q)}_{n=odd=1} \quad = \quad & -\frac{1}{2}\left(\frac{q}{p}\right)\frac{\pi}{\sin(\pi q/p)}\left\{\left[2\psi\left(\frac{q}{p}\right)+\frac{1}{(q/p)}-2\psi(1)-2+2\ln 2\right]\right. \\ & \left. +\cos(\pi q/p)\left[\psi\left(\frac{q}{2p}+\frac{1}{2}\right)-\psi\left(\frac{q}{2p}\right)-\frac{1}{(q/p)}\right]\right\} \end{aligned} \tag{135}$$

To unify notation we redefine the previous two sums by

$$\tilde{S}{}^{u(q,p)}_{n} = \left\{ \begin{matrix} \tilde{E}{}^{u(q,p)}_{n} \quad for\ n = 2k+1 > 0 \\ \tilde{O}{}^{u(q,p)}_{n} \quad for\ n = 2k \geqslant 0 \end{matrix} \right\} \tag{136}$$

and

$$\overline{S}{}^{u(q,p)}_{n} = \left\{ \begin{matrix} E_n^{u(q,p)} for\ n = 2k+1 > 0 \\ O_n^{u(q,p)} for\ n = 2k \geqslant 0 \end{matrix} \right\} \tag{137}$$

Again, an exercise in contour integration, generating functions, differential equations and difference equations, leads to

$$\begin{aligned} \widetilde{S}_n^{u(q,p)} \quad = \quad & \left[\widetilde{S}_1^{u(q/p)} - \frac{2q}{p}\frac{\pi}{2}\frac{1}{\sin(\pi q/p)}\right]\frac{p}{2q}u_n^{q/p} - \frac{\pi}{2\sin(\pi q/p)}\sum_{k=1}^{n}\frac{(-)^k}{k}u_k^{q/p}u_{n-k}^{q/p} \\ & -\frac{\pi}{2}\tan\left(\pi\frac{q}{2p}\right)\frac{p}{2q}\sum_{k=0}^{n-1}\frac{(-)^k}{k+1}u_k^{q/p}u_{n-1-k}^{q/p} \end{aligned} \tag{138}$$

This result holds for all integer values of $n \geq 1$. For negative values of n, one finds

$$\widetilde{S}_{-n}^{u(q,p)} = \cos\left(\pi\frac{q}{p}\right)\widetilde{S}_n^{u(q,p)} + \left[1+\cos\left(\pi\frac{q}{p}\right)\right]\overline{S}_0^{u(q,p)}S_n^{u(q,p)},\ n > 0 \tag{139}$$

$$\overline{S}_0^{u(q,p)} = \frac{1}{2}\pi\tan\left(\pi\frac{q}{2p}\right)$$

Detailed treatment of these sums as well as the derivation of the G matrix are given in reference [15].

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