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Abstract

In this paper we construct the operator connecting the three vertex in Split string field theory and the dual model vertex of Sciuto, Caneshi, Schwimmer and Veneziano (SCSV). This construction results in an explicit conformal transformation linking the two interactions in the matter sector at all levels. Thus establishing the equivalence between the two theories at least for $N = 3$. Furthermore the construction of the conformal operator leads to important identities between the infinite dimensional change of representation matrices between the Split field theory and Witten's string theory. These identities turn proved to be an essential tool in establishing the Bose-Fermi equivalence of the two theories.

Contents

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1 Introduction

The recent work of references [25, 26, 27, 28] has generated much interest in the comma formulation of Witten's theory of interacting open bosonic strings. An important role in the formulation of the comma theory is played by the $BRST$ charge Q of the first quantized theory [13, 15]. In general, the $BRST$ invariance of the first quantized theory becomes a gauge invariance of the second theory [34, 16, 7, 17]. In the interacting comma theory, the role of the ghost fields is quite subtle. The rich structure of the ghost sector of the interacting theory deserves more consideration. The comma theory is intimately related to the conformally invariant $2-$dimensional field theory. It is possible to study the ghosts using either bosons or fermions. However, to relate the fermionic ghost formulation to the bosonic ghost formulation, we have to carry a bosonization procedure explicitly [7, 31, 32, 33]. The bosonic and the fermionic realization of the ghost fields for the comma theory have been used in ref. [12] and [13] respectively, to write down the ghost part of the three-comma vertex and the proof of equivalence was only addressed partially in [33]. Although both formulations give a gauge invariant theory, their equivalence is not at all transparent. It is the purpose of this paper to complete the proof of equivalence for both formulations. The key to the proof as we have seen in the first part of the proof [33] lies on the various identities satisfied by the G-coefficients that define the comma interacting vertex. To complete the proof of equivalence in [33], we have to show that both realizations of the comma vertex satisfy the same Ward-like identities.

Like in the matter sector for the comma formulation of Witten's theory for open bosonic strings, in the half-string approach to the ghost part of string theory, the elements of the theory are defined by $\delta - function$ type overlaps

$$V_3^{\phi} = \exp\left(i\sum_{r=1}^{3} Q_r^{\phi}\phi\left(\pi/2\right)\right) V_{3,0}^{\phi} \tag{1}$$

where

$$V_{3,0}^{\phi}=\prod_{r=1}^{3}\prod_{\sigma=0}^{\pi/2}\delta\left(\phi_r^L\left(\sigma\right)-\phi_{r-1}^R\left(\sigma\right)\right) \tag{2}$$

and the half string ghost coordinates defined in [12, 33], $\phi_r^{L,R}\left(\sigma\right)$, $r=1,2,3$, are given by

$$\begin{aligned}\phi_r^L\left(\sigma\right) &= \phi_r(\sigma)-\phi_r(\frac{\pi}{2}), \quad \sigma\in\left[0,\frac{\pi}{2}\right]\\ \phi_r^R\left(\sigma\right) &= \phi_r(\pi-\sigma)-\phi_r(\frac{\pi}{2}), \quad \sigma\in\left[0,\frac{\pi}{2}\right]\end{aligned} \tag{3}$$

where $\phi_r(\sigma)$ is the full string coordinate

$$\phi_r\left(\sigma\right)=\phi_0^r+\sqrt{2}\sum_{n=1}^{\infty}\phi_n^r\cos n\sigma, \quad \sigma\in[0,\pi] \tag{4}$$

The index r refers to the rth string (it is to be understood that $r-1=0\equiv 3$). The factor Q_r^{ϕ} is the ghost number insertion at the mid-point which is needed for the $BRST$ invariance of the theory [18, 12] and in this case $Q_1^{\phi}=Q_2^{\phi}=Q_1^3=1/2$. As we have seen before in the Hilbert space of the theory, the $\delta-functions$ translate into operator overlap equations which determine the precise form of the vertex. The ghost part of the comma vertex in the full string basis has the same structure as the coordinate one apart from the mid-point insertions

$$|V_{\phi}^{HS}>=e^{\frac{1}{2}i\sum_{r=1}^{3}\phi^r(\pi/2)}V_{\phi}^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)\left|0,N_{ghost}=\frac{3}{2}\right)_{123}^{\phi} \tag{5}$$

where the α's are the bosonic oscillators defined by the expansion of the bosonized ghost $\left(\phi\left(\sigma\right),p^{\phi}\left(\sigma\right)\right)$ fields and $V_{\phi}^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right)$ is the exponential of the quadratic form in the ghost creation operators with the same structure as the coordinate piece of the vertex

$$\begin{aligned}V_{\phi}^{HS}\left(\alpha^{\phi,1\dagger},\alpha^{\phi,2\dagger},\alpha^{\phi,3\dagger}\right) &= \exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m=1}^{\infty}\alpha_{-n}^{\phi,r}G_{nm}^{rs}\alpha_{-m}^{\phi,s}+\right.\\ &\left.\sum_{r,s=1}^{3}p_0^{\phi,r}G_{0m}^{rs}\alpha_{-m}^{\phi,s}+\frac{1}{2}\sum_{r,s=1}^{3}p^{\phi,r}G_{00}^{rs}p_0^{\phi,s}\right]\end{aligned} \tag{6}$$

where the the matrix elements G_{nm}^{rs} have been constructed in [?].

In the full string, the fermionic ghost overlap equations are

$$\begin{aligned}c_r\left(\sigma\right) &= c_{r-1}(\pi-\sigma) \quad \sigma\in\left[0,\frac{\pi}{2}\right]\\ c_r\left(\sigma\right) &= -c_{r+1}(\pi-\sigma)\ , \quad \sigma\in\left[\frac{\pi}{2},\pi\right]\end{aligned} \tag{7}$$

and

$$
\begin{aligned}
b_r(\sigma) &= b_{r-1}(\pi-\sigma) \quad \sigma \in \left[0, \frac{\pi}{2}\right] \\
b_r(\sigma) &= b_{r+1}(\pi-\sigma) \quad , \quad \sigma \in \left[\frac{\pi}{2}, \pi\right]
\end{aligned} \tag{8}
$$

The proof of the bose-fermi equivalence involves two major obstacles. The first is to show that the bosonized half string ghosts, (5), satisfy the $c-$ and $b-overlap$ equations displayed above. To carry out the proof, in [33], the authors utilized the bosonization formulas

$$
c_+(\sigma) =: e^{i\phi_+(\sigma)} : , \qquad b_+(\sigma) =: e^{-i\phi_-(\sigma)} : \tag{9}
$$

where $\phi(\sigma) = \frac{1}{2}\left(\phi_+(\sigma) + \phi_-(\sigma)\right)$, and

$$
\phi_\pm(\sigma) = \phi_0 \pm \sigma\left(p_0^\phi + \frac{1}{2}\right) + i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_n^\phi e^{\mp in\sigma} - \alpha_{-n}^\phi e^{\pm in\sigma}\right) \tag{10}
$$

The fermionic ghost coordinates of the bosonic string are anticommuting fields

$$
\begin{aligned}
c_\pm(\sigma) &= c(\sigma) \pm i\pi_b(\sigma) \quad \sigma \in [0, \pi] \\
b\pm(\sigma) &= \pi_c(\sigma) \pm ib(\sigma) \quad , \quad \sigma \in [0, \pi]
\end{aligned} \tag{11}
$$

The $c_+ (c_-)$ are the ghosts for reparametrization of $z = \tau + i\sigma$ ($\overline{z} = \tau - i\sigma$) respectively and the $b_\pm$ are the corresponding anti-ghosts. These obey the anticommutation relations

$$
\begin{aligned}
\{c_n, c_m\} &= \{b_n, b_m\} = 0 \\
\{c_n, b_m\} &= \delta_{n+m\ 0}
\end{aligned} \tag{12}
$$

The fermionic half string ghosts were defined in [13] by

$$
\begin{aligned}
c_r^L(\sigma) &= c_r(\sigma) - c_r(\frac{\pi}{2}), \quad \sigma \in \left[0, \frac{\pi}{2}\right] \\
c_r^R(\sigma) &= c_r(\pi-\sigma) - c_r(\frac{\pi}{2}), \quad \sigma \in \left[0, \frac{\pi}{2}\right]
\end{aligned} \tag{13}
$$

and likewise for $b^L(\sigma)$ and $b^R(\sigma)$.

2 Half-string Coordinates

Here we are going to give a brief derivation of the transformation matrices between the half string coordinates and the full string coordinates needed for the construction of the half-string interacting vertex in terms of the oscillator representation of the full-string. For this we shall follow closely the discussion of reference [11, 12, 13, 14, 15]. The standard mode expansion for each of the 26 coordinates of the open bosonic string in Fourier modes are given by

$$
x^\mu(\sigma) = x_0^\mu + \sqrt{2}\sum_{n=1}^{\infty} x_n^\mu \cos(n\sigma), \qquad \sigma \in \left[0, \frac{\pi}{2}\right] \tag{14}
$$

To simplify the notation, henceforth, we shall drop the spatial indices. The modes in (14) can be related to creation and annihilation operators in the usual way

$$a_n = -i\sqrt{\frac{n}{2}}x_n + \frac{1}{\sqrt{2n}}p_n, \quad a_n^\dagger = i\sqrt{\frac{n}{2}}x_n + \frac{1}{\sqrt{2n}}p_n \tag{15}$$

for $n > 0$, where $p_n = -i\frac{\partial}{\partial x_n}$. For the zero modes they are related by

$$a_0 = -ix_0 + \frac{1}{2}p_0, \quad a_0^\dagger = ix_0 + \frac{1}{2}p_n \tag{16}$$

$p_0 = -i\frac{\partial}{\partial x_0}$. The creation and annhiliation operators satisfy the commutation relations

$$\left[a_n, a_m^\dagger\right] = \delta_{nm} \tag{17}$$

The half-string coordinates $x^L(\sigma)$ and $x^R(\sigma)$ for the left and right half of the string are defined in the usual way

$$x^L(\sigma) = x(\sigma) - x(\frac{\pi}{2}), \quad x^R(\sigma) = x(\pi - \sigma) - x(\frac{\pi}{2}), \quad 0 \leq \sigma \leq \frac{\pi}{2} \tag{18}$$

where both $x^L(\sigma)$ and $x^R(\sigma)$ satisfy the usual Neumann boundary conditions at $\sigma = 0$ and a Dirichlet boundary conditions $\sigma = \pi/2$. Hence, their mode expansion takes the form

$$x^r(\sigma) = \sqrt{2}\sum_{n=1}^{\infty} x_n^r \cos(n\sigma) \tag{19}$$

where $r = 1, 2$, refers to the left (L) and right (R) halves of the string, respectively. Comparing equation (14) and equation (19) we can relate the half-string modes to the full-string modes

$$x_n^r = (-1)^{r+1} x_{2n-1} + \sum_{m=1}^{\infty} \left(\frac{2m}{2n-1}\right)^{1/2} \left[M_{mn}^1 + M_{mn}^2\right] x_{2m} \tag{20}$$

$n = 1, 2, 3, ...$ The M^1 and M^2 are the change of representation matrices

$$M_{m\ n}^r = \frac{2}{\pi}\left(\frac{2m}{2n-1}\right)^{1/2} \frac{(-1)^{m+n}}{2m + (-1)^r (2n-1)}, \quad m, n > 0 \tag{21}$$

Since the transformation in (20) is non singular, we can invert these relation to obtain

$$\begin{aligned} x_{2n-1} &= \frac{1}{2}\left(x_n^L - x_n^R\right), \\ x_{2n} &= \frac{1}{2}\sum_{m=1}^{\infty}\left(\frac{2m-1}{2n}\right)^{1/2}\left[M_{mn}^1 - M_{mn}^2\right]\left(x_m^L + x_m^R\right) \end{aligned} \tag{22}$$

where $n = 1, 2, 3, ...$, and

$$x_0 = x_M - \frac{\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \left(x_n^L + x_n^R \right) \tag{23}$$

Using the commutation relations

$$[x_n^r, p_m^s] = i\delta^{rs}\delta_{nm} \tag{24}$$

we can derive the relationships between the half-string conjugate momenta and the full-string conjugate momenta and

$$p_{_M} = p_0 \tag{25}$$

Theses relations can be inverted to give

$$\begin{aligned} p_{2n-1} &= p_n^L - p_n^R, \\ p_{2n} &= \sum_{m=1}^{\infty} \left(\frac{2n}{2m-1} \right)^{1/2} \left[M_{nm}^1 + M_{nm}^2 \right] \left(p_m^L + p_m^R \right) + \sqrt{2} \left(-1 \right)^n p_M \end{aligned} \tag{26}$$

where $n > 0$. We notice that the existence of the one-to-one correspondence between the half string and the full string degrees of freedom guarantees the existence of the identification

$$\overline{H} = \overline{H_M \otimes H_L \otimes H_R} \tag{27}$$

where $\overline{H}$ stands for the completion of the full string Hilbert space and H_L, H_R, H_M in the tensor product stand for the two half-string Hilbert spaces and the Hilbert space of functions of the mid-point, respectively.

3 The Half-String Overlaps

The half string three interaction of the open bosonic string (V_x^{HS}) has been derived in ref. INSERT. Here we are interested in expressing the comma three interaction vertex in terms of the full-string mode basis. This form of the half-string three vertex is more suitable for realting the V_x^{HS} to the dual model vertex of Sciuto, Caneschi, Schwimmer and Veneziano, V_x^{SCSV} [21]. Here we shall only consider the coordinate piece of the comma three interaction vertex. The ghost part of the vertex (V_ϕ^{HS}) in the bosonic representation is identical to the coordinate piece apart from the ghost mid-point insertions $3i\phi\left(\pi/2\right)/2$ required for ghost number conservation at the mid-point. To simplify the calculation we introduce a new set of coordinates and momenta based on a Z_3 Fourier transform[1]

$$Q_j^r(\sigma) = \frac{1}{\sqrt{3}} \sum_{k=1}^{3} \chi_k^r(\sigma)\, e^{2\pi ijk/3} \tag{28}$$

[1] This technique was first used by D. Gross and A. Jevicki in 1986.

with identical expression for $Þ_j^r(\sigma)$. Notice that in the Z_3 Fourier space the commutation relations are

$$\left[Q^r(\sigma), \overline{Þ}^s(\sigma')\right] = \left[\overline{Q}^r(\sigma), Þ^s(\sigma')\right] = \left[Q_3^r(\sigma), Þ_3^s(\sigma')\right] = i\delta^{rs}\delta(\sigma - \sigma')$$

where we have denoted $Q^r(\sigma) \equiv Q_1^r(\sigma) = \overline{Q}_2^r(\sigma)$ and $Þ^r(\sigma) \equiv Þ_1^r(\sigma) = \overline{Þ}_2^r(\sigma)$. So $Q^r(\sigma)$ and $Þ^r(\sigma)$ are no longer canonical variables. The canonical variables in this case are $Q^r(\sigma)$ and $\overline{Þ}^r(\sigma)$. The variables $Q_3^r(\sigma)$ and $Þ_3^r(\sigma')$ are still canonical however. The overlap equations for the comma coordinates

$$\begin{aligned} \chi_j^r(\sigma) &= \chi_{j-1}^{r-1}(\sigma), \qquad 0 \leq \sigma \leq \pi/2 &(29)\\ x_M^1 &= x_M^2 = x_M^3 &(30)\end{aligned}$$

where $x_M \equiv x(\pi/2)$ and the identification $r-1=0\equiv 2$ and $j-1=0\equiv 3$ is understood, and the definition in (28) now yields

$$\begin{aligned} Q^L(\sigma) &= eQ^R(\sigma), \qquad 0 \leq \sigma \leq \pi/2 &(31)\\ \overline{Q}^L(\sigma) &= \bar{e}\overline{Q}^R(\sigma), \qquad 0 \leq \sigma \leq \pi/2 &(32)\\ Q_M &= \overline{Q}_M = 0 &(33)\\ Q_3^L(\sigma) &= Q_3^R(\sigma), \qquad 0 \leq \sigma \leq \pi/2 &(34)\\ Q_{3,M} &= Q_{3,M} &(35)\end{aligned}$$

for the overlap equations in the Z_3-Fourier space. The equation (33) is to be understood as a constraint equation. The overlaps for the canonical momenta

$$\begin{aligned} \wp_j^r(\sigma) &= -\wp_{j-1}^{r-1}(\sigma), \qquad 0 \leq \sigma \leq \pi/2 &(36)\\ 0 &= \wp_{1,M} + \wp_{2,M} + \wp_{3,M} &(37)\end{aligned}$$

where the mid-point momentum is defined in the usual way $\wp_M \equiv -i\partial/\partial x_M = -i\partial/\partial x_0 = p_0$, in the Z_3-Fourier space of the comma become

$$\begin{aligned} Þ^L(\sigma) &= -eÞ^R(\sigma), \qquad 0 \leq \sigma \leq \pi/2 &(38)\\ \overline{Þ}^L(\sigma) &= -\bar{e}\overline{Þ}^R(\sigma), \qquad 0 \leq \sigma \leq \pi/2 &(39)\\ Þ_3^L(\sigma) &= -Þ_3^R(\sigma), \qquad 0 \leq \sigma \leq \pi/2 &(40)\\ P_{3,M} &= 0 &(41)\end{aligned}$$

The overlap conditions on $Q^r(\sigma)$ and $Þ^r(\sigma)$ determine the form of the comma three interaction vertex. In terms of the Fourier modes, the overlap equations translate to

$$\begin{aligned} Q_{2n-1}^L &= eQ_{2n-1}^R, \quad n = 1, 2, 3, ... &(42)\\ \overline{Q}_{2n-1}^L &= \bar{e}\overline{Q}_{2n-1}^R, \quad n = 1, 2, 3, ... &(43)\\ Q_M &= \overline{Q}_M = 0 &(44)\end{aligned}$$

$$Q^L_{3,2n-1} = Q^R_{3,2n-1}, \quad n=1,2,3,... \tag{45}$$
$$Q_{3,M} = Q_{3,M} \tag{46}$$

for the coordinates and

$$Þ^L_{2n-1} = -eÞ^R_{2n-1}, \quad n=1,2,3,... \tag{47}$$
$$\overline{Þ}^L_{2n-1} = -\bar{e}\overline{Þ}^R_{2n-1}, \quad n=1,2,3,... \tag{48}$$

$$Þ^L_{3,2n-1} = -Þ^R_{3,2n-1}, \quad n=1,2,3,... \tag{49}$$
$$P_{3,M} = 0 \tag{50}$$

for their conjugate momenta. The corresponding creation and annihilation operators $\left(B^r_n, B^r_{-n}\right)$, where the convention $B^r_{-n} = \overline{B}^{r\dagger}_n$ and $\overline{B}^r_{-n} = B^{r\dagger}_n$ is adopted, are related to the half-string creation and annhilition opertors (b^r_n, b^r_{-n}) through (28) and satisfy the commutation relations

$$\left[B^r_n, \overline{B}^s_m\right] = \delta^{rs}\delta_{n+m,0}, \quad n,m \neq 0$$

In the Z_3-Fourier space of the comma the overlap equations separate into two sets. The comma vertex $V^{HS}_x\left(b^{1,r\dagger}, b^{2,r\dagger}, b^{3,r\dagger}\right)$, therefore separates into a product of two pieces one depending on $B^{3,r\dagger}$and the other one depending on $\left(B^{r\dagger}, \overline{B^{r\dagger}}\right)$. Observe that the first of these equations is identical to the overlap equation for the identity vertex. Hence, the comma 3-Vertex takes the form

$$|V^{HS}_Q > = \int dQ_M d\overline{Q}_M dQ^3_M \delta\left(Q_M\right)\delta\left(\overline{Q}_M\right) e^{iP^3_M Q^3_M} \times \exp\left[-\frac{1}{2}\left(B^{3\dagger}\left|C\right|B^{3\dagger}\right) - \left(B^\dagger\left|H\right|\overline{B^\dagger}\right)\right]\prod_{r=L,R}|0>^{3,r}|0>^r|\overline{0}>^r$$

where C and H are infinite dimensional matrices computed in [23] and the integration over Q^3_M gives $\delta\left(P^3_M\right)$. However $P^3_M = P^3_0$ (see ref. Bordes) and so $\delta\left(P^3_M\right)$ is the statements of conservation of momentum at the center of mass of the three string. In the usual space of the string, the above expression reduces to

$$|V^{HS}_x > = \int \prod_{i=1}^{3} dx^i_M \delta\left(x^i_M - x^{i-1}_M\right)\delta\left(\sum_{j=1}^{3} p^j_M\right) \times \exp\left[-\sum_{j=1}^{3}\sum_{n=1}^{\infty} b^{j,L\dagger}_n b^{j-1,R\dagger}_n\right]\prod_{j=1}^{3}|0>^L_j |0>^R_j \tag{51}$$

Here $b_n^{j,L(R)}$ denotes oscillators in the $L\,(R)$ jth string Fock space. Once again for simplicity the Lorentz index ($\mu = 0, 1, ..., 25$) and the Minkowski metric $\eta_{\mu\nu}$ used to contract the Lorentz indices, have been suppressed in equation (51). We shall follow this convention throughout this paper.

Though the form of the comma 3-Vertex given in equation (51) is quite elegant, it is very cumbersome to relate it directly to the $SCSV$ 3-Vertex due to the fact that connection between the vacuum in the comma theory and the vacuum in the $SCSV$ is quite involved. One also needs to use the change of representation formulas [11] to recast the quadratic form in the half string creation operators in terms of the full string creation-annihilation operators which adds more complications to an already difficult problem. Alternatively we could rewrite the $SCSV$ vertex in the comma basis. Both ways are involved and lead to a considerable amount of algebra. On the other hand the task could be greatly simplified if we express the comma vertex in the full string basis. This may be achieved simply by re expressing the comma overlaps in terms of overlaps in the full string basis.

4 The Half-String 3-Vertex in the Full-String basis

The relationship between the half-string Z_3-Fourier modes and the full string Z_3-Fourier modes, can be derived from (20), (**??**) and the definition in (28) to give

$$Q_n^r = (-1)^{r+1} Q_{2n-1} + \sum_{m=1}^{\infty} \left(\frac{2m}{2n-1} \right)^{1/2} \left[M^1_{m\;n} + M^2_{m\;n} \right] Q_{2m} \quad (52)$$

$$\text{Þ}_n^r = \frac{(-1)^{r+1}}{2} P_{2n-1} + \frac{1}{2} \sum_{m=1}^{\infty} \left(\frac{2n-1}{2m} \right)^{1/2} \left[M^1_{m\;n} - M^2_{m\;n} \right] P_{2m} - \frac{\sqrt{2}}{\pi} \frac{(-1)^n}{2n-1} P_0$$

where $n = 1, 2, 3, ...$ The overlap equations in (42), (47) and (33) now takes the form

$$(1+e)\, Q_{2n-1} = -(1-e) \sum_{m=1}^{\infty} \left(\frac{2m}{2n-1} \right)^{1/2} \left[M^1_{m\;n} + M^2_{m\;n} \right] Q_{2m} \quad (53)$$

$$(1-e)\, \frac{1}{2} P_{2n-1} = -(1+e)\, \frac{1}{2} \sum_{m=1}^{\infty} \left(\frac{2n-1}{2m} \right)^{1/2} \left[M^1_{m\;n} - M^2_{m\;n} \right] P_{2m} + (1+e)\, \frac{\sqrt{2}}{\pi} \frac{(-1)^n}{2n-1} P_0 \quad (54)$$

$$Q_M = Q_0 + \sqrt{2} \sum_{n=1}^{\infty} (-1)^n Q_{2n} = 0 \quad (55)$$

respectively. The overlaps for the complex conjugate of the first two equations could be obtained simply by taking the complex conjugation. Similarly from the overlaps in (45), (49) and (41) we obtain

$$Q^3_{2n-1} = 0 \quad (56)$$

$$\sum_{m=1}^{\infty}\left(\frac{2n-1}{2m}\right)^{1/2}\left[M^1_{m\ n} - M^2_{m\ n}\right]P^3_{2m} - \frac{2\sqrt{2}}{\pi}\frac{(-1)^n}{2n-1}P^3_0 = 0 \quad (57)$$

$$Þ^3_M = 0 \quad (58)$$

We have seen in reference [11] the $Þ^3_M = P^3_0$ and so the overlap conditions in (57) and (58) reduce to

$$\sum_{m=1}^{\infty}\left(\frac{2n-1}{2m}\right)^{1/2}\left[M^1_{m\ n} - M^2_{m\ n}\right]P^3_{2m} = 0 \quad (59)$$

$$P^3_0 = 0 \quad (60)$$

Equation (60) is the statement of the conservation of the momentum carried by the third string in the Z_3 Fourier space.

Because of the decoupling of the degrees of freedom in the Z_3-Fourier space, the half-string 3-vertex in the full string basis, $|V^{HS}\left(A^3_{-n},\overline{A}_{-n},A_{-n}\right)>$, seperates into a product of two pieces depending on A^3_{-n} and on $\left(\overline{A}_{-n},A_{-n}\right)$ respectively, where $A^3_{-n} = A^{3\dagger}_n$, $A_{-n} = \overline{A}^{\dagger}_n$, $\overline{A}_{-n} = A^{\dagger}_n$ and $n > 0$. For the matter sector, the half-string 3-vertex in the full string Z_3-Fourier space takes the form

$$|V^{HS}_Q >= \int dQ^3_M dQ_M d\overline{Q}_M \delta\left(Q_M\right)\delta\left(\overline{Q}_M\right)e^{iP^3_0 Q^3_M}V^{HS}\left(\left(A^3_{-n},\overline{A}_{-n},A_{-n}\right)\right)|0)_{123} \quad (61)$$

where $|0)_{123}$ denotes the matter part of the vacuum in the Hilbert space of the three strings and

$$V^{HS}\left(A^{3\dagger}_n, A^{\dagger}_n, \overline{A}^{\dagger}_n\right) = \exp\left[-\frac{1}{2}\sum_{n,m=0}^{\infty}A^{3\dagger}_n C_{nm}A^{3\dagger}_m - \sum_{n,m=0}^{\infty}A^{\dagger}_n F_{nm}\overline{A}^{\dagger}_m\right] \quad (62)$$

The half-string 3-vertex, $|V^{HS}\left(A^3_{-n},\overline{A}_{-n},A_{-n}\right)>$, satisfies the comma overlaps in (53), The overlaps equations in (56) and (59) are identical to the overlap equations for the identity vertex [18, 19, 14, 15]. Hence,

$$C_{nm} = (-1)^n\,\delta_{nm}, \qquad n,m = 0,1,2,... \quad (63)$$

The explicit form of the matrix F, may be obtained from the overlap equations given by (**??**), (**??**) and (**??**) as well as their complex conjugates. It will turn out that the matrix F has the following properties

$$F = F^{\dagger}, \quad \overline{F} = CFC, \quad F^2 = 1 \quad (64)$$

which are consistent with the properties of the coupling matrices in Witten's theory of open bosonic strings [18, 19]. This indeed is a nontrivial check on the validity of the comma approach to the theory of open bosonic strings.

Next substituting (62) into (**??**) and writing Q_n in terms of $A_n^\dagger$ and A_n, we obtain the first equation for the matrix F

$$F_{2n-1\ k} + \delta_{2n-1\ k} - i\sqrt{3}\sum_{m=1}^{\infty}\left(M^1_{m\ n} + M^2_{m\ n}\right)\left(F_{2m\ k} + \delta_{2m\ k}\right) = 0 \qquad (65)$$

where $k = 0, 1, 2..., n = 1, 2, 3, ...$ Next from the overlap equation in (**??**) we obtain a second condition on the F matrix

$$\begin{aligned} 0 &= \left(F_{2n-1\ k} - \delta_{2n-1\ k}\right) + \frac{1}{\sqrt{3}} i \sum_{m=1}^{\infty}\left(M^1_{m\ n} - M^2_{m\ n}\right)\left(F_{2m\ k} - \delta_{2m\ k}\right) \\ &\quad - \frac{4}{\pi}\frac{i}{\sqrt{3}}\frac{(-1)^n}{(2n-1)^{3/2}}\left(F_{0\ k} - \delta_{0\ k}\right) \end{aligned} \qquad (66)$$

where $k = 0, 1, 2..., n = 1, 2, 3, ...$ The overlaps for the mid-point in (**??**) give

$$\left[\left(F_{0m} + \delta_{0m}\right) + \sqrt{2}\sum_{k=1}^{\infty}(-1)^k\sqrt{\frac{2}{2k}}\left(F_{2km} + \delta_{2k\ m}\right)\right] = 0, \quad m = 0, 1, 2, ... \qquad (67)$$

Solving equations (65) and (66), we have (Be careful in the equations below; I have relabeled the indices on the matrices just so the indices are what is usually used to label matrix elements)

$$F_{2n\ 0} = \frac{1}{\pi}\left(F_{00} - 1\right)\sum_{m=1}^{\infty}\left[\left(M_1^T + \frac{1}{2}M_2^T\right)^{-1}\right]_{nm}\frac{(-)^m}{(2m-1)^{3/2}} \qquad (68)$$

$$\begin{aligned} F_{2n\ 2k} &= \frac{1}{\pi}F_{0\ 2k}\sum_{m=1}^{\infty}\left[\left(M_1^T + \frac{1}{2}M_2^T\right)^{-1}\right]_{n\ m}\frac{(-)^m}{(2m-1)^{3/2}} \\ &\quad - \sum_{m=1}^{\infty}\left[\left(M_1^T + \frac{1}{2}M_2^T\right)^{-1}\right]_{n\ m}\left[\frac{1}{2}M_1^T + M_2^T\right]_{m\ k} \end{aligned} \qquad (69)$$

$$\begin{aligned} F_{2n\ 2k-1} &= -\frac{i\sqrt{3}}{2}\left[\left(M_1^T + \frac{1}{2}M_2^T\right)^{-1}\right]_{n\ k} + \frac{1}{\pi}F_{0\ 2k-1} \\ &\quad \times\sum_{m=1}^{\infty}\left[\left(M_1^T + \frac{1}{2}M_2^T\right)^{-1}\right]_{n\ m}\frac{(-)^m}{(2m-1)^{3/2}} \end{aligned} \qquad (70)$$

$$F_{2n-1\ 2k-1} = \frac{2i}{\sqrt{3}}\sum_{m=1}^{\infty}\left[\frac{1}{2}M_1^T + M_2^T\right]_{n\ m} F_{2m\ 2k-1} + \frac{2i}{\pi\sqrt{3}}\frac{(-)^n}{(2n-1)^{3/2}}F_{0\ 2k-1} \qquad (71)$$

$$F_{2n-1\ 2k} = \frac{2i}{\sqrt{3}} \sum_{m=1}^{\infty} \left[\frac{1}{2}M_1^T + M_2^T\right]_{n\ m} F_{2m\ 2k} + \frac{2i}{\sqrt{3}} \left[M_1^T + \frac{1}{2}M_2^T\right]_{nk} + \frac{2i}{\pi\sqrt{3}} \frac{(-)^n}{(2n-1)^{3/2}} F_{0\ 2k} \tag{72}$$

$$F_{2n-1\ 0} = \frac{2i}{\sqrt{3}} \sum_{m=1}^{\infty} \left[\frac{1}{2}M_1^T + M_2^T\right]_{n\ m} F_{2m\ 0} + \frac{2i}{\pi\sqrt{3}} \frac{(-)^n}{(2n-1)^{3/2}} (F_{0\ 0} - 1) \tag{73}$$

where all $n, k = 1, 2, 3, ...$Finally equation (67) leads to

$$(F_{00} + 1) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2n}} F_{2n\ 0}, \tag{74}$$

$$F_{0\ 2m} = 2\frac{(-1)^{m+1}}{\sqrt{2m}} + 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k}} F_{2k\ 2m}, \tag{75}$$

$$F_{0\ 2m-1} = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k}} F_{2k\ 2m-1} \tag{76}$$

where $m = 1, 2, 3, ...$

Now the explicit form of the F matrix is completely given by the set of equations (68), (69), (70), (71), (72), (73), (74), (75) and (76) since the inverse of the $\left(M_1^T + \frac{1}{2}M_2^T\right)$ exists. The inverse of the $\left(M_1^T + \frac{1}{2}M_2^T\right)$ has been evaluated in refrence [?]

$$\left[\left(M_1^T + \frac{1}{2}M_2^T\right)^{-1}\right]_{nm} = \frac{1}{\sqrt{3}}(-)^{n+m} (2n)^{1/2} (2m-1)^{1/2} \left[\frac{a_{2n}b_{2m-1} + b_{2n}a_{2m-1}}{2n - (2m-1)} + \frac{a_{2n}b_{2m-1} - b_{2n}a_{2m-1}}{2n + (2m-1)}\right] \tag{77}$$

where the coefficients a_k and b_k are the modes appearing in the Taylor expansion of the functions $\left(\frac{1+x}{1-x}\right)^{1/3}$ and $\left(\frac{1+x}{1-x}\right)^{2/3}$ respectively.

5 Computing the explicit values of the matrix elements of the F matrix

Here we shall give the steps involved in the computation of the matrix elements of F; the technical details can be found in [?]. For the purpose of illustration consider F_{2n0}. Substituting the explicit value of $\left(M_1^T + \frac{1}{2}M_2^T\right)^{-1}$obtained in (77) into equation (68) gives

$$F_{2n\ 0} = \frac{1}{\pi} (F_{00} - 1) \frac{1}{\sqrt{3}} (-)^n (2n)^{1/2} \frac{1}{2n} [2(a_{2n}) O_0^b - a_{2n}O_{-2n}^b - b_{2n}O_{-2n}^a - a_{2n}O_{2n}^b + b_{2n}O_{2n}^a] \tag{78}$$

where the quantities appearing in the above expression are defined by

$$O_0^b = \sum_{m=1}^{\infty} \frac{b_{2m-1}}{2m-1}, \quad O_{\pm 2n}^a = \sum_{m=1}^{\infty} \frac{a_{2m-1}}{\pm 2n + (2m-1)}, \quad O_{\pm 2n}^b = \sum_{m=1}^{\infty} \frac{b_{2m-1}}{\pm 2n + (2m-1)}$$

and have been evaluated in reference [18, 19, ?]. The specific value of O_0^b has been given in (??); the values of O_{2n}^a, O_{-2n}^a, O_{2n}^b and O_{-2n}^b are given in equations (??), (??), (??) and (??), respectively. Thus substituting the explicit values of these quantities into (78) and combining terms we find

$$F_{2n\ 0} = (F_{00} - 1)\, \frac{(-)^n a_{2n}}{\sqrt{2n}} \tag{79}$$

The explicit value of the F_{00} may be computed by substituting (79) into (74). Hence,

$$\frac{1 + F_{00}}{1 - F_{00}} = \ln \frac{3^3}{2^4} \tag{80}$$

which gives the explicit value of F_{00} at once. This result is consistent with that given in [18, 19]. To obtain the explicit value of $F_{0\ 2m}$, we first need to evaluate the sum over k in equation (75); that is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k}} F_{2k\ 2m}$$

where the explicit expression for $F_{2k\ 2m}$ in terms of the change of representation matrices is given by equation (69). Thus substituting (69) into the above expression we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k}} F_{2k\ 2m} &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k}} \frac{1}{\pi} F_{0\ 2m} \sum_{l=1}^{\infty} \left[\left(M_1^T + \frac{1}{2} M_2^T \right)^{-1} \right]_{k\ l} \frac{(-)^l}{(2l-1)^{3/2}} \\ &- \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k}} \sum_{l=1}^{\infty} \left[\left(M_1^T + \frac{1}{2} M_2^T \right)^{-1} \right]_{k\ l} \left[\frac{1}{2} M_1^T + M_2^T \right]_{lm} \end{aligned} \tag{81}$$

If we commute[2] the sums over k and l, and sum over k we find

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k}} \left[\left(M_1^T + \frac{1}{2} M_2^T \right)^{-1} \right]_{k\ l} = -\frac{2}{\sqrt{3}} \frac{(-)^l a_{2l-1}}{(2l-1)^{1/2}} \tag{82}$$

Now substituting equation (82) into (81), then using the explicit value of M_1 and M_2 and rewriting $\ln\left(3^3/2^4\right)$ in terms of F_{00}, we get

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k}} F_{2k\ 2m} = -\frac{1}{2} \left(\frac{1 + F_{0\ 0}}{1 - F_{0\ 0}} \right) F_{0\ 2m} + \frac{(-1)^m (1 - a_{2m})}{(2m)^{1/2}}$$

[2] Since both the sums over l and k are uniformly convergent, one may perform the sums in any order. We have carried the sums in the two different orders and found that the result is the same. However, it is much easier to perform the sum over k first followed by the sum over l rather than the reverse. Here we shall follow the former.

Substituting this result into (75), we find

$$F_{0\ 2m} = (F_{00} - 1) \frac{(-1)^m a_{2m}}{(2m)^{1/2}} \tag{83}$$

which has the same form as $F_{2m\ 0}$ given in (79). Thus in this case we see that the property $F_{even\ 0} = \left(F^\dagger\right)_{0\ even}$ holds.

Next we consider the evaluation of $F_{2n-1\ 0}$. If we replace M_1, M_2 and $F_{2m\ 0}$ in (73) by their explicit values, given respectively by equation (**??**), (**??**) and (79), we have

$$\begin{aligned} F_{2n-1\ 0} &= \frac{2i}{\sqrt{3}} \frac{2}{\pi} (F_{00} - 1) \frac{(-1)^n}{\sqrt{2n-1}} \left[\frac{1}{2} \frac{a_0}{(2n-1)} + \frac{1}{2} \sum_{m=0}^{\infty} \frac{a_{2m}}{2m - (2n-1)} \right. \\ & \left. - \frac{a_0}{(2n-1)} + \sum_{m=0}^{\infty} \frac{a_{2m}}{2m + (2n-1)} + \frac{2i}{\pi\sqrt{3}} \frac{(-)^n}{(2n-1)^{3/2}} (F_{0\ 0} - 1) \right] \end{aligned} \tag{84}$$

The sums in the square brackets have been evaluated in [**?**] and their explicit values are given by equations (**??**) and (**??**), respectively. Thus substituting equations (**??**) and (**??**) in equation (84) and collecting like terms, one finds

$$F_{2n-1\ 0} = i (F_{00} - 1) \frac{(-1)^n a_{2n-1}}{\sqrt{2n-1}}, \quad n = 1, 2, 3, ... \tag{85}$$

To check if the property $F = F^\dagger$ continue to hold, we need to compute explicitly the value of $F_{0\ 2n-1}$. It is important to verify that the matrix F is self adjoint for the consistency of our formulation. The matrix element $F_{0\ 2n-1}$ involves the matrix element $F_{2n\ 2k-1}$ which in turn is expressed in terms of the combination $\left(M_1^T + \frac{1}{2} M_2^T\right)^{-1}$ and the matrix element $F_{0\ 2n-1}$ itself. To carry out the calculation, unfortunately we first need to compute the explicit value of $F_{2n\ 2k-1}$. Putting the explicit value of $\left(M_1^T + \frac{1}{2} M_2^T\right)^{-1}$ into (70) and rearranging terms we get

$$\begin{aligned} F_{2n\ 2k-1} &= \frac{(-)^{n+k}}{2i} \sqrt{2n}\sqrt{2k-1} \left[\frac{a_{2n} b_{2k-1} + b_{2n} a_{2k-1}}{2n - (2k-1)} + \frac{a_{2n} b_{2k-1} - b_{2n} a_{2k-1}}{2n + (2k-1)} \right] \\ & + \frac{(-)^n a_{2n}}{\sqrt{2n}} F_{0\ 2k-1}, \qquad n = 1, 2, 3, .. \end{aligned} \tag{86}$$

Combining equation (86) with equation (76), leads to

$$F_{0\ 2m-1} = -i (F_{00} - 1) (-)^m \frac{a_{2m-1}}{\sqrt{2m-1}} \tag{87}$$

which is precisely the adjoint of $F_{2m-1\ 0}$; see equation (85). Thus we have

$$F_{0\ odd} = \left(F^\dagger\right)_{0\ odd} \tag{88}$$

as expected.

The result obtained in (87) may be now used to find the explicit value of $F_{2n\ 2m-1}$. Thus substituting equation (87) back into equation (86), we find

$$F_{2n\ 2m-1} = \frac{(-)^{n+m}}{2i}\sqrt{2n}\sqrt{2m-1}\left[\frac{a_{2n}b_{2m-1}+b_{2n}a_{2m-1}}{2n-(2m-1)}+\frac{a_{2n}b_{2m-1}-b_{2n}a_{2m-1}}{2n+(2m-1)}\right] -i\left(F_{00}-1\right)\frac{(-)^{n+m}a_{2n}a_{2m-1}}{\sqrt{2n}\sqrt{2m-1}}, \quad n,m=1,2,3,.. \tag{89}$$

The computation of the matrix element $F_{2n-1\ 2m}$ is indeed quite cumbersome. The difficulty arises from the fact that the defining equation of $F_{2n-1\ 2m}$, which is given by (72), involves this summing over the matrix $F_{2m\ 2k}$ which is potentially divergent when the summing index m takes the k value. The limiting procedures involved in smoothing out the divergence are quite delicate and require careful consideration. Thus here we shall only give the final result, however the derivation is given in appendix INSERT,

$$F_{2n-1\ 2m} = -\frac{(-)^{n+m}}{2i}\sqrt{2m}\sqrt{2n-1}\left[\frac{a_{2m}b_{2n-1}+b_{2m}a_{2n-1}}{2m-(2n-1)}+\frac{a_{2m}b_{2n-1}-b_{2m}a_{2n-1}}{2m+(2n-1)}\right] +i\left(F_{00}-1\right)\frac{(-)^{n+m}a_{2m}a_{2n-1}}{\sqrt{2m}\sqrt{2n-1}} \tag{90}$$

Comparing equations (89) and (90), we see that

$$F_{even\ odd} = \left(F^{\dagger}\right)_{even\ odd} \tag{91}$$

as expected.

To complete fixing the comma interaction vertex in the full-string basis we still need to compute the remaining elements; namely $F_{2n\ 2m}$ and $F_{2n-1\ 2m-1}$. The computing of the matrices $F_{2n\ 2m}$ and $F_{2n-1\ 2m-1}$ involve two distinct cases. The off diagonal case is given by $n \neq m$ and the diagonal case is given by $n = m$. Though the off diagonal elements are not difficult to compute, the diagonal elements are indeed quite involved and they can be evaluated by setting $n = m$ in the defining equations for $F_{2n\ 2m}$ and $F_{2n-1\ 2m-1}$ and then explicitly performing the sums with the help of the various identities we have established in ref. [**?**]. An alternative way of computing the diagonal elements is to take the limit of $n \to m$ in the explicit expressions for the off diagonal elements. We have computed diagonal elements both ways and obtained the same result which is a non trivial consistency check on our formulation. For illustration, here we shall compute the diagonal elements by the limiting process we spoke of as we shall see shortly. But first let us compute the off diagonal elements. We first consider $F_{2n\ 2m}$. Putting the explicit value of $\left(M_1^T + \frac{1}{2}M_2^T\right)^{-1}$ and $\frac{1}{2}M_1^T + M_2^T$

into (69), we find

$$F_{2n\ 2k} = F_{0\ 2k}\frac{(-)^n a_{2n}}{(2n)^{1/2}} - \frac{(-)^{n+k}(2k)^{1/2}(2n)^{1/2}}{2}\frac{a_{2n}b_{2k}+b_{2n}a_{2k}}{2n+2k} - \frac{2}{\pi}\frac{(-)^{n+k}}{\sqrt{3}}(2k)^{1/2}(2n)^{1/2}$$
$$\times\left\{\frac{1}{2}\sum_{m=1}^{\infty}(a_{2n}b_{2m-1}+b_{2n}a_{2m-1})\left(\frac{1}{2n-(2m-1)}\frac{1}{2k-(2m-1)}\right)\right.$$
$$\left.+\sum_{m=1}^{\infty}(a_{2n}b_{2m-1}-b_{2n}a_{2m-1})\left(\frac{1}{2n+(2m-1)}\frac{1}{2k+(2m-1)}\right)\right\} \tag{92}$$

The difficulty in evaluating the sums arises from the fact in performing these sums one usually make use of partial fraction to reduce them to the standard sums treated in appendix A; however partial fraction in this case fails due to a divergence arising from the particular case when $n = m$. Thus to carry our program through, we first consider the case for which $n \neq k$. For $n \neq k$, partial fraction can be used to reduce the sums in the above expression to the standard results obtained in appendix A. Skipping some rather straight forward algebra, we find

$$F_{2n\ 2k} = (F_{00}-1)\frac{(-)^{n+k}a_{2n}a_{2k}}{(2n)^{1/2}(2k)^{1/2}} - \frac{(-)^{n+k}(2k)^{1/2}(2n)^{1/2}}{2}$$
$$\times\left[\frac{a_{2n}b_{2k}+b_{2n}a_{2k}}{2n+2k}+\frac{a_{2n}b_{2k}-b_{2n}a_{2k}}{2n-2k}\right],\quad n,k=1,2,...$$

valid for $n \neq k$. Note that in this case we have

$$F_{even\ even} = \left(F^{\dagger}\right)_{even\ even} \tag{93}$$

as expected. As we pointed earlier the diagonal element $F_{2n\ 2k}$ may be obtained by taking the limit of $k \to n$ in equation (92); hence

$$F_{2n\ 2n} = \lim_{k\to n} F_{2n\ 2k} = F_{0\ 2n}\frac{(-)^n a_{2n}}{(2n)^{1/2}} - \frac{a_{2n}b_{2n}+b_{2n}a_{2n}}{4}$$
$$-\frac{2}{\pi}\frac{2n}{\sqrt{3}}\left[\frac{1}{2}a_{2n}\widetilde{S}^b_{-2n}+\frac{1}{2}b_{2n}\widetilde{S}^a_{-2n}+a_{2n}\widetilde{S}^b_{2n}-b_{2n}\widetilde{S}^a_{2n}\right]$$

This result may be simplified further with the help of the following identities derived in (**??**) and (**??**)

$$\widetilde{S}^a_{-2n} = \frac{1}{2}\widetilde{S}^a_{2n} + \frac{1}{4}\pi\sqrt{3}S^a_{2n}\ ,\ n>0$$

and

$$\widetilde{S}^b_{-2n} = -\frac{1}{2}\widetilde{S}^b_{2n} + \frac{1}{4}\pi\sqrt{3}S^b_{2n}\ ,\ n>0$$

$$b_{2n} S_{2n}^{a} + a_{2n} S_{2n}^{b} = \frac{2}{2n}$$

which are derived in ref. in [?] and equation (83) to eliminate $F_{0\ 2n}$, we obtain

$$F_{2n\ 2n} = (F_{00} - 1) \frac{a_{2n} a_{2n}}{2n} - \frac{1}{2} b_{2n} a_{2n} - \frac{1}{2} - \frac{2n}{\pi} \frac{\sqrt{3}}{2} \left(a_{2n} \widetilde{S}_{2n}^{b} - b_{2n} \widetilde{S}_{2n}^{a} \right) \quad (94)$$

Hence,

$$F_{even\ even} = \left(F^{\dagger}\right)_{even\ even}$$

as expected. Finally we consider the matrix elements $F_{odd\ odd}$. If we put the values of $F_{2m\ 2k-1}$ and $F_{0\ 2k-1}$ are given by equations (89) and (87) respectively, in equation (71), and skipping some rather straightforward algebra, we find

$$\begin{aligned} F_{2n-1\ 2k-1} \quad = \quad & (F_{00} - 1) \frac{(-1)^{n+k}}{3} \frac{a_{2k-1} a_{2n-1}}{(2n-1)^{1/2} (2k-1)^{1/2}} + \frac{(-)^{k+n}}{2} (2n-1)^{1/2} (2k-1)^{1/2} \\ & \times \frac{a_{2n-1} b_{2k-1} + b_{2n-1} a_{2k-1}}{(2n-1) + (2k-1)} + \frac{2}{\sqrt{3}} \frac{(-)^{k+n} \sqrt{2k-1} \sqrt{2n-1}}{\pi} \\ & \left\{ \frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{2m - (2n-1)} \frac{1}{2m - (2k-1)} (a_{2m} b_{2k-1} + b_{2m} a_{2k-1}) \right. \\ & \left. - \sum_{m=0}^{\infty} \frac{1}{2m + (2n-1)} \frac{1}{2m + (2k-1)} (a_{2m} b_{2k-1} - b_{2m} a_{2k-1}) \right\} \end{aligned} \quad (95)$$

Now there are two cases to consider $k \neq n$ and $k = n$. For $k \neq n$, equation (95) becomes

$$\begin{aligned} F_{2n-1\ 2k-1} \quad = \quad & (F_{00} - 1) \frac{(-1)^{n+k}}{3} \frac{a_{2k-1} a_{2n-1}}{(2n-1)^{1/2} (2k-1)^{1/2}} + \frac{(-)^{k+n}}{2} (2n-1)^{1/2} (2k-1)^{1/2} \\ & \times \left[\frac{a_{2n-1} b_{2k-1} + b_{2n-1} a_{2k-1}}{(2n-1) + (2k-1)} + \frac{a_{2n-1} b_{2k-1} - b_{2n-1} a_{2k-1}}{(2n-1) - (2k-1)} \right] \end{aligned} \quad (96)$$

where $n, k = 1, 2, 3$. Thus for $n \neq k$, we see that

$$F_{odd\ odd} = \left(F^{\dagger}\right)_{odd\ odd}, \qquad \text{for } n \neq k$$

For $k = n$, equation (95) becomes

$$\begin{aligned} F_{2n-1\ 2n-1} \quad = \quad & \frac{1}{3} (F_{00} - 1) \frac{a_{2n-1} a_{2n-1}}{(2n-1)^{1/2} (2n-1)^{1/2}} + \frac{1}{2} a_{2n-1} b_{2n-1} \\ & + \frac{2}{\sqrt{3}} \frac{(2n-1)}{\pi} \left\{ \frac{1}{2} \left[b_{2n-1} \widetilde{E}_{-(2n-1)}^{a} + a_{2n-1} \widetilde{E}_{-(2n-1)}^{b} \right] \right. \\ & \left. - \left[b_{2n-1} \widetilde{E}_{(2n-1)}^{a} - a_{2n-1} \widetilde{E}_{(2n-1)}^{b} \right] \right\} \end{aligned} \quad (97)$$

Using the identities

$$\begin{aligned}\widetilde{E}^a_{-(2n-1)} &= \frac{1}{2}\widetilde{E}^a_{2n-1} + \frac{1}{4}\pi\sqrt{3}S^a_{2n-1} \\ \widetilde{E}^b_{-(2n-1)} &= -\frac{1}{2}\widetilde{E}^b_{2n-1} + \frac{1}{4}\pi\sqrt{3}S^b_{2n-1}\end{aligned}$$

derived in [**?**], the above expression becomes

$$\begin{aligned}F_{2n-1\ 2n-1} &= \frac{1}{3}\left(F_{00}-1\right)\frac{a_{2n-1}a_{2n-1}}{(2n-1)^{1/2}\,(2n-1)^{1/2}} + \frac{1}{2}a_{2n-1}b_{2n-1} + \frac{1}{2} \\ &\quad + \frac{\sqrt{3}}{2\pi}\,(2n-1)\left(a_{2n-1}\widetilde{E}^b_{(2n-1)} - b_{2n-1}\widetilde{E}^a_{(2n-1)}\right) \qquad (98)\end{aligned}$$

which is clearly self adjoint. Thus from equations (96) and (98) it follows that

$$F_{odd\ \ odd} = \left(F^{\dagger}\right)_{odd\ \ odd} \qquad (99)$$

as expected. With this result we, establish that $F = F^{\dagger}$ as anticipated.

In the original variables, the comma three-string in (61) can be written in the form

$$|V_x^{HS}> = \delta\left(\sum_{r=1}^{3} p_0^r\right) e^{-\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m=0}^{\infty} a^r_{-n}\mathcal{F}^{rs}_{nm}a^s_{-m}}|0>_{123} \qquad (100)$$

The matrix elements $\mathcal{F}^{ij}_{nm}$ may be obtained by comparing (100) to (61) . Skipping the details, which are given in ref. [**?**], we find

$$\begin{aligned}\mathcal{F} &= \frac{1}{3}\left[\left(C+F+\overline{F}\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \left(C-\frac{F+\overline{F}}{2}\right)\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\right. \\ &\quad \left. + i\frac{\sqrt{3}}{2}\left(F-\overline{F}\right)\begin{pmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{pmatrix}\right] \qquad (101)\end{aligned}$$

Equation (101) gives completely the half-string interaction 3-vertex in the full string basis in the representation with oscillator zero modes.

Sometimes it is useful to express the comma vertex in the momentum representation. For a single oscillator with momentum p and creation operator $\alpha^{\dagger}$, the change of basis is accomplished by

$$|p> = \exp\left(-\frac{1}{2}p_0\overline{p}_0 + \overline{p}_0\alpha_0^{\dagger} + \overline{\alpha}_0^{\dagger}p_0 - \alpha^{\dagger}\overline{\alpha}^{\dagger}\right)|0>$$

with $|0>$ being the oscillator ground state. Thus using the above identity and equation (62) one finds the following representation for the Vertex in the momentum space

$$\exp\left[-\frac{1}{2}\sum_{n,m=0}^{\infty} A_n^{3\dagger}C_{nm}A_m^{3\dagger}-\sum_{n,m=1}^{\infty} A_n^{\dagger}F'_{nm}\overline{A}_m^{\dagger}\right.$$
$$\left.-\sum_{n=0}^{\infty}A_n^{\dagger}F'_{n0}\overline{P}_0-\sum_{n=0}^{\infty}P_0F'_{0m}\overline{A}_m^{\dagger}+\frac{1}{2}\overline{P}_0F'_{00}P_0\right] \qquad (102)$$

where the prime matrices F'_{nm} are related to the unprimed matrices F'_{nm} by

$$F'_{00}=\frac{1+F_{00}}{1-F_{00}},\ \ F'_{0n}=\frac{F_{0n}}{1-F_{00}},\ \ F'_{nm}=F_{nm}+\frac{F_{n0}F_{0m}}{1-F_{00}} \qquad (103)$$

where $n,m=1,2,3,....$ The property $F^2=1$ in equation (64) implies that in the momentum representation, the F' matrix satisfies

$$\sum_{k=1}^{\infty}F'_{nk}F'_{km}=\delta_{nm},\quad n,m=1,2,3,... \qquad (104)$$

For $n\neq 0$, we have $a^r_{-n}\equiv\alpha^r_{-n}/\sqrt{n}$, and so equation (102) may be written as

$$|V_x^{HS}\quad > \ =\int\prod p_0^r\exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m=1}^{\infty}\alpha^r_{-n}G^{rs}_{nm}\alpha^s_{-m}+\sum_{r,s=1}^{3}p_0^rG^{rs}_{0m}\alpha^s_{-m}\right.$$
$$\left.+\frac{1}{2}\sum_{r,s=1}^{3}p^rG^{rs}_{00}p^s_0\right]|0,p)_{123} \qquad (105)$$

where the matrix G is defined through the relation

$$G^{rs}_{nm}=-\frac{1}{\sqrt{n+\delta_{n0}}}\mathcal{F}'^{rs}_{nm}\frac{1}{\sqrt{m+\delta_{m0}}} \qquad (106)$$

The matrix elements G^{rs}_{nm} are listed in appendix B. The properties of the G matrix can be found from the identities in (104). The G matrix satisfies the important identity.

$$\sum_{t=1}^{3}\sum_{k=1}^{\infty}kG^{rt}_{nk}G^{ts}_{km}=\frac{1}{n}\delta^{rs}\delta_{nm} \qquad (107)$$

We are now in a position to construct the operator connecting the coordinate part of the comma vertex and the coordinate part of Caneschi-Schwimmer-Veneziano vertex of the dual model.

6 The Operator Connecting the SCSV 3-Vertex and the Comma 3-Vertex

In [24], the explicit operator connecting the covariant and the dual vertices was constructed. The existence of the conformal operator was ensured by the fact that the construction of Witten's covariant string theory was related to the dual model through a conformal mapping. The existence of the transformations, guarantees that all physical couplings of the vertex operators are identical, and moreover gives an alternative nontrivial computational tool as it has been shown in [24]. We have seen in [23, 14, 15, 25, 26, 27, 28], that the comma theory offers an alternative way of formulating Witten's covariant theory in terms of the half-string degrees of freedom. The equivalence between the comma theory and Witten's theory was discussed in [23, 14, 15], where most of the technical details were overcome, but for a few conceptual points regarding the uniqueness of the Witten interaction, and the role played by the mid-point of the string. To help understand these delicate points further, it is important that we find a way of relating the comma theory to the dual model of Sciuto, Caneshi, Schwimmer and Veneziano [21, 22]. In this section, we shall construct explicitly an operator connecting the comma theory and the dual model. The general procedure described here follows closely that employed in [24]. Furthermore here will only concentrate on the matter part of the vertex, and a similar approach will be used for the ghost part of the vertex, which will be presented in [29].

The Sciuto-Caneschi-Schwimmer-Veneziano vertex (hereafter, referred to as the SCSV Vertex) has the explicit form [21, 22]

$$< V_x^{SCSV}| =< 0,0,0|\delta\left(\sum_{r=1}^{3} p_0^r\right)\exp\left[\frac{1}{2}\sum_{i,j=1}^{3}\sum_{n=1,m=0}^{\infty}\alpha_n^i M_{nm}^{ij}\alpha_m^j\right] \tag{108}$$

where

$$M_{nm}^{12} = M_{nm}^{23} = M_{nm}^{31} = \frac{(-1)^m}{n}\begin{pmatrix} n \\ m \end{pmatrix} \tag{109}$$

and all other M's vanish. The form of the vertex in (108) is equivalent to the overlap equations

$$< V_x^{SCSV}|\left[\alpha_{-n}^r - \sum_{s=1}^{3}\sum_{m=0}^{\infty} nM_{nm}^{rs}\alpha_m^s\right] = 0 \tag{110}$$

where $r = 1,2,3$ and $n = 1,2,3,....$ Likewise, from equation (100), we can derive the equation

$$< V_x^{HS}|\left[\alpha_{-n}^r - \sum_{s=1}^{3}\sum_{m=0}^{\infty} nG_{nm}^{rs}\alpha_m^s\right] = 0 \tag{111}$$

where $r = 1,2,3$ and $n = 1,2,3,...$ Here the string indices $r,s = 1,2,3$ and the mode indices $n,m = 0,1,2,....$ It is important to notice that the relations in (111), are equivalent to the overlap equations for defining the comma three vertex in the full string basis in the sense that they determine the matter part of the vertex, but they are more convenient than the form in (108) for our purpose here.

Unlike the case of the Witten's covariant theory and the dual model where the fact that both of the two vertices give the same coupling for all physical on-shell states,

$$(L_n^r - \delta_{n0})\,|phys >= 0, \;\; n = 0, 1, 2, ..., \tag{112}$$

guarantee the existence of an operator [24, 30]

$$O = \exp \sum_{r=1}^{3} \sum_{n=0}^{\infty} A_n^r \left(L_n^r - \delta_{n0}\right) \tag{113}$$

such that

$$< V_x^{SCSV}|O =< V_x^{HS}|, \tag{114}$$

In the case of the comma theory, we have to prove the existence of the operator connecting the comma vertex and the SCSV vertex by explicit construction. We will look for an operator of the form in (113)

$$\hat{O} = \exp \sum_{r=1}^{3} \sum_{n=0}^{\infty} \mathbf{\Lambda}_n^r \left(L_n^r - \delta_{n0}\right) \tag{115}$$

such that

$$< V_x^{SCSV}|\hat{O} =< V_x^{HS}| \tag{116}$$

Equation (110) now gives

$$0 =< V_x^{HS}|\hat{O}^{-1} \left[\alpha_{-n}^r - \sum_{s=1}^{3} \sum_{m=0}^{\infty} n M_{nm}^{rs} \alpha_m^s\right] \hat{O} \tag{117}$$

The completeness of (111), grantees that this is some linear combination of relations (111) for different values of the indices r and n. To determine the values of the coefficients $\mathbf{\Lambda}_n^r$, we need to compare (111) and (117). To do this successfully, we first need to compute the basic commutator

$$\left[\alpha_n^r, e^{\sum_{m=0}^{\infty} \mathbf{\Lambda}_m^r (L_m^r - \delta_{m\;0})}\right] \tag{118}$$

The Virasoro generators are given in terms of the α's by

$$L_m^r = \frac{1}{2} \sum_{k=-\infty}^{\infty} : \alpha_{m-k}^r \; \alpha_k^r : \tag{119}$$

where, the normal ordering : : is with respect to the full string vacuum. The commutator in (118) is some function of the commutators $[L_m^r, \alpha_n^r]$, commutators of these commutators, etc. Thus equation (118) gives[3]

$$\left[\alpha_n^r, e^{\xi \sum_{m=0}^{\infty} \mathbf{\Lambda}_m^r L_m^r}\right] = e^{\xi \sum_{m=0}^{\infty} \mathbf{\Lambda}_m^r L_m^r} \sum_k f_k^{(r,n)}(\xi)\, \alpha_k^r \tag{120}$$

[3] We have ignored the constant term $\mathbf{\Lambda}_n^s \delta_{n0}$ in the exponential $\sum_{n=0}^{\infty} \mathbf{\Lambda}_n^s \left(L_n^s - \delta_{n0}\right)$ since this piece is a $C - number$ and so it commutes with the α's.

where we have introduced a parameter ξ. Notice that this result reduces to the commutator in (118) by setting $\xi = 1$. Our original conformal operator in (115) is related to $\hat{O}(\xi)$ through the relation $\hat{O} = \hat{O}(\xi)|_{\xi=1}$. By differentiating with respect to this parameter, one obtains a set of differential equations for the functions $f_k^{(r,n)}(\xi)$ which can be solved exactly as we shall see shortly[4]. Differentiating both sides of the above expression and use the commutation relation

$$[L_m^r, \alpha_n^r] = -m\alpha_{n+m}^r \tag{121}$$

to pull the L's past the α's, we find

$$\sum_k \frac{df_k^{(r,n)}(\xi)}{d\xi}\alpha_k^r = \sum_k k f_k^{(r,n)}(\xi) \sum_{m=0}^{\infty} \mathbf{\Lambda}_m^r \alpha_{m+k}^r + \sum_{m=0}^{\infty} n\mathbf{\Lambda}_m^r \alpha_{m+n}^r \tag{122}$$

If we now write $\alpha_{m+k}^r = \sum_l \alpha_l^r\ \delta_{l\ m+k}$ and $\alpha_{m+n}^r = \sum_k \alpha_k^r\ \delta_{k\ m+n}$ and exchange the dummy indices l and k as needed, the above expression becomes

$$\sum_k \frac{df_k^{(r,n)}(\xi)}{d\xi}\alpha_k^r = \sum_k \left[\sum_{m=0}^{\infty} (k-m)\,\mathbf{\Lambda}_m^r f_{k-m}^{(r,n)}(\xi) + \sum_{m=0}^{\infty} (k-m)\,\mathbf{\Lambda}_m^r \delta_{k-m\ n}\right]\alpha_k^r \tag{123}$$

Since the α's are all linearly independent, it follows that

$$\frac{df_k^{(r,n)}(\xi)}{d\xi} = \sum_{m=0}^{\infty} (k-m)\,\mathbf{\Lambda}_m^r \left[f_{k-m}^{(r,n)}(\xi) + \delta_{k-m\ n}\right] \tag{124}$$

To solve the system of differential equations, we need the boundary condition for each of the k functions. This we accomplish by setting $\xi = 0$ in equation (120).

$$[\alpha_n^r, 1] = \sum_k f_k^{(r,n)}(0)\,\alpha_k^r \tag{125}$$

Since the commutator on the left hand side is identically zero and the α's are all linearly independent, it follows that

$$f_k^{(r,n)}(0) = 0 \tag{126}$$

for all values of k. The set of k equations in (126) are the desired k boundary conditions. Using (126) in (124), we find that

$$\left.\frac{df_k^{(r,n)}(\xi)}{d\xi}\right|_{\xi=0} = 0 \tag{127}$$

[4]In fact the function $f_k^{(r,n)}(\xi)$ does not depend on the string index r due to the cyclic symmetry in the string indices. Moreover it does not depend on the particular mode n and so we could drop the string label and the mode label if please. However, I have decided to keep them here for bookkeeping.

for all values of $k < n$. Combining equations (126) and (127), we find that

$$f_k^{(r,n)}(\xi) = 0 \tag{128}$$

for all values of $k < n$. For $k \geq n$, we make the substitution $k = n + q$, $q = 0, 1, 2, ...$ in equation (124)

$$\frac{df_k^{(r,n)}(\xi)}{d\xi} = \sum_{m=0}^{\infty} (n + q - m) \boldsymbol{\Lambda}_m^r \left[f_{n+q-m}^{(r,n)}(\xi) + \delta_{m\ q} \right] \tag{129}$$

Since $f_k^{(r,n)}(\xi) = 0$ for $k < n$, then the infinite sum over the first term on the right hand side of the above expression reduces to a finite sum, that is, $\sum_{m=0}^{\infty} (n + q - m) \Lambda_m^r f_{n+q-m}^{(r,n)}(\xi) = \sum_{k=0}^{q} (n + k) \Lambda_{q-k}^r f_{n+k}^{(r,n)}(\xi)$. Likewise the second infinite sum over the second term on the right hand side reduces to a finite sum, that is, $\sum_{m=0}^{\infty} (n + q - m) \boldsymbol{\Lambda}_m^r \delta_{m\ q} = \sum_{k=0}^{q} (n + k) \Lambda_{q-k}^r \delta_{k\ 0}$. Thus the above expression reduces to

$$\frac{df_{n+q}^{(r,n)}(\xi)}{d\xi} = \sum_{k=0}^{q} (n + k) \Lambda_{q-k}^r \left[f_{n+k}^{(r,n)}(\xi) + \delta_{k\ 0} \right] \tag{130}$$

Making the substitution $\widetilde{f}_{n+m}^{(r,n)}(\xi) = f_{n+m}^{(r,n)}(\xi) + \delta_{m\ 0}$, the differential equation takes on a more elegant form

$$\frac{d\widetilde{f}_{n+q}^{(r,n)}(\xi)}{d\xi} = \sum_{k=0}^{q} (n + k) \Lambda_{q-k}^r \widetilde{f}_{n+k}^{(r,n)}(\xi) \tag{131}$$

To find the explicit form of $\widetilde{f}_{n+q}^{(r,n)}(\xi)$, we need to solve equation (131) for all values of q. This can be simply achieved by solving (131) for the first few values of q and then guessing the general form of the solution or by mathematical induction. Therefore, let us first consider $q = 0$, so that equation (131) becomes

$$\frac{d\widetilde{f}_n^{(r,n)}(\xi)}{d\xi} = n\Lambda_0^r \widetilde{f}_n^{(r,n)}(\xi) \tag{132}$$

which has the solution

$$\widetilde{f}_n^{(r,n)}(\xi) = C_0 e^{n\Lambda_0^r \xi} \tag{133}$$

where C_0 is the constant of integration. Next we consider $q = 1$. Setting $q = 1$ in equation (131) and then eliminating $\widetilde{f}_n^{(r,n)}(\xi)$ with the help of (133), we find

$$\frac{d\widetilde{f}_{n+1}^{(r,n)}(\xi)}{d\xi} = n\Lambda_1^r C_0 e^{n\Lambda_0^r \xi} + (n + 1) \Lambda_0^r \widetilde{f}_{n+1}^{(r,n)}(\xi) \tag{134}$$

which has the well known solution

$$\widetilde{f}_{n+1}^{(r,n)}(\xi) = e^{(n+1)\Lambda_0^r \xi} \int^{\xi} n\Lambda_1^r C_0 e^{-\Lambda_0^r \xi} d\xi + C_1 e^{(n+1)\Lambda_0^r \xi} \tag{135}$$

where C_1 is the constant of integration. Evaluating a rather simple integral, we obtain

$$\widetilde{f}_{n+1}^{(r,n)}(\xi) = -C_0 n \left(\frac{\Lambda_1^r}{\Lambda_0^r}\right) e^{n\Lambda_0^r \xi} + C_1 e^{(n+1)\Lambda_0^r \xi} \tag{136}$$

At this point, it is not hard to see that the general solution has the form

$$\widetilde{f}_{n+q}^{(r,n)}(\xi) = \sum_{k=0}^{q} C_{qk}^{[r,n]} e^{(n+k)\Lambda_0^r \xi} \tag{137}$$

To determine the coefficients $C_{qk}^{[r,n]}$, we only need to substitute this result back in (131); doing so we obtain

$$\sum_{l=0}^{q} C_{ql}^{[r,n]} e^{(n+l)\Lambda_0^r \xi} (n+l)\Lambda_0^r = \sum_{k=0}^{q} (n+k)\Lambda_{q-k}^r \sum_{l=0}^{k} C_{kl}^{[r,n]} e^{(n+l)\Lambda_0^r \xi} \tag{138}$$

We notice that the diagonal part of the coefficient $C_{ql}^{[r,n]}$, that is, $C_{qq}^{[r,n]}$ has the same multiplicative factor on both sides of the equation and so it drops out and the above expression after a bit of rather straight forward algebra reduces to

$$\sum_{k=0}^{q-1} C_{qk}^{[r,n]} e^{(n+k)\Lambda_0^r \xi} (k-q)\Lambda_0^r = \sum_{l=0}^{q-1}\sum_{k=0}^{l} (n+l)\Lambda_{q-l}^r C_{lk}^{[r,n]} e^{(n+k)\Lambda_0^r \xi} \tag{139}$$

In obtaining the above results, we had to exchange the dummy indices $k \leftrightarrows l$ at some point of the calculation. If we expand the right hand side, we see at once that the double sum $\sum_{l=0}^{q-1}\sum_{k=0}^{l}(\cdots) = \sum_{k=0}^{q-1}\sum_{l=k}^{q-1}(\cdots)$, and so one obtains a recursion relation between the $C_{qk}^{[r,n]}$ coefficients

$$C_{qk}^{[r,n]}(k-q)\Lambda_0^r = \sum_{l=k}^{q-1}(n+l)\Lambda_{q-l}^r C_{lk}^{[r,n]} \tag{140}$$

This result gives the off diagonal elements of $C_{qk}^{[r,n]}$ in terms of the $C_{kk}^{[r,n]}$, $C_{k+1k}^{[r,n]}$, $C_{k+2k}^{[r,n]}$, ..., $C_{q-1k}^{[r,n]}$. Thus we have

$$C_{qk}^{[r,n]} = -\frac{1}{q-k}\sum_{l=k}^{q-1}(n+l)\, C_{lk}^{[r,n]}\overline{\Lambda}_{q-l}^r \tag{141}$$

where $\overline{\Lambda}_m^r \equiv \Lambda_m^r/\Lambda_0^r$. To obtain the diagonal elements of $C_{qk}^{[r,n]}$, one needs to set $\xi = 0$ in equation (137) for $q = 0, 1, 2, ...$, and use the fact that $\widetilde{f}_{n+q}^{(r,n)}(0) = \delta_{q\,0}$

(which follows from equation (126) and the definition of $\widetilde{f}_{n+q}^{(r,n)}(\xi)$), to find

$$\begin{aligned} C_{00}^{[r,n]} &= 1 \\ C_{10}^{[r,n]} + C_{11}^{[r,n]} &= 0 \\ C_{20}^{[r,n]} + C_{21}^{[r,n]} + C_{22}^{[r,n]} &= 0 \\ &. \\ &. \\ &. \\ C_{q0}^{[r,n]} + C_{q1}^{[r,n]} + \cdots + C_{qq-1}^{[r,n]} + C_{qq}^{[r,n]} &= 0 \end{aligned}$$

which can be written in a more compact form, that is

$$C_{qq}^{[r,n]} = \delta_{q0} - \sum_{k=0}^{q-1} C_{qk}^{[r,n]}, \quad q = 0, 1, 2, ... \tag{142}$$

To evaluate equation (117), we need to move the operator $\hat{O}$ to the left side of the square bracket on the right hand side of equation (117). To do this successfully, we first need to compute the action of α on $\hat{O}$, that is we need to compute $\alpha_n^r \hat{O}$. Consider

$$\alpha_n^r \hat{O} = \hat{O} \alpha_n^r + \left[\alpha_n^r, \hat{O}\right] \tag{143}$$

Using equation (120) to evaluate the commutator, the above expression becomes

$$\alpha_n^r \hat{O} = \hat{O} \left(\alpha_n^r + \sum_k f_k^{(r,n)}(\xi)\,|_{\xi=1} \alpha_k^r \right) \tag{144}$$

Using the fact that

$$f_k^{(r,n)}(1) = \begin{cases} 0, & k < n \\ f_{n+q}^{(r,n)}(1), & k = n+q,\ q = 0, 1, 2, ... \end{cases} \tag{145}$$

then the sum in equation (144) becomes $\sum_{q=0}^{\infty} f_{n+q}^{(r,n)}(1)\, \alpha_{n+q}^r$ and so equation (144) takes the form

$$\alpha_n^r \hat{O} = \hat{O} \left(\alpha_n^r + \sum_{q=0}^{\infty} f_{n+q}^{(r,n)}(1)\, \alpha_{n+q}^r \right) \tag{146}$$

If we now recall that $\widetilde{f}_{n+q}^{(r,n)}(1) = f_{n+q}^{(r,n)}(1) + \delta_{q\,0}$, then the above equation becomes

$$\alpha_n^r \hat{O} = \hat{O} \sum_{q=0}^{\infty} \widetilde{f}_{n+q}^{(r,n)}(1)\, \alpha_{n+q}^r \tag{147}$$

Furthermore, since $\widetilde{f}_n^{(r,n)}(1) = C_{00}^{[r,n]} e^{n\Lambda_0^r}$ and $C_{00}^{[r,n]} = 1$, we find

$$\alpha_n^r \hat{O} = \hat{O}\left(e^{n\Lambda_0^r}\alpha_n^r + \sum_{q=1}^{\infty} \widetilde{f}_{n+q}^{(r,n)}(1)\,\alpha_{n+q}^r\right) \tag{148}$$

Replacing $\widetilde{f}_{n+q}^{(r,n)}(1)$ by its value from equation (137), the above expression becomes

$$\alpha_n^r \hat{O} = \hat{O} e^{n\Lambda_0^r}\left(\alpha_n^r + \sum_{q=1}^{\infty}\sum_{k=0}^{q} C_{qk}^{[r,n]} e^{k\Lambda_0^r}\alpha_{n+q}^r\right) \tag{149}$$

and so the desired identity for $\hat{O}^{-1}\alpha_n^r\hat{O}$ follows at once

$$\hat{O}^{-1}\alpha_n^r \hat{O} = e^{n\Lambda_0^r}\left(\alpha_n^r + \sum_{q=1}^{\infty}\sum_{k=0}^{q} C_{qk}^{[r,n]} e^{k\Lambda_0^r}\alpha_{n+q}^r\right) \tag{150}$$

This result is valid for all integral values of n including zero. Using this result in (117), we find

$$\begin{aligned} 0 =< V_x^{HS}| \Bigg[& \alpha_{-n}^r + \sum_{q=1}^{\infty}\sum_{k=0}^{q} C_{qk}^{[-n]} e^{k\Lambda_0}\alpha_{-n+q}^r \\ & - \sum_{s=1}^{3}\sum_{m=0}^{\infty} n M_{nm}^{rs} e^{(m+n)\Lambda_0}\left(\alpha_m^s + \sum_{q=1}^{\infty}\sum_{k=0}^{q} C_{qk}^{[m]} e^{k\Lambda_0}\alpha_{m+q}^s\right)\Bigg] \end{aligned} \tag{151}$$

Where we have dropped the superscript from the Λ_k^r and $C_{qk}^{[r,n]}$ constants which follow from the observation that the $G's$ and $M's$ matrices have cyclic symmetry in the string indices and thus $\Lambda_k \equiv \Lambda_k^1 = \Lambda_k^2 = \Lambda_k^3$ and $C_{qk}^{[n]} \equiv C_{qk}^{[1,n]} = C_{qk}^{[2,n]} = C_{qk}^{[3,n]}$.

We notice that this result is a linear combination of the first n equations in (111). To fix the Λ's, we need to compare the coefficients of the operators α's in equation (151) to those in equation (111) for all the n modes. The simplest relation we can retrieve from equations (111) and (151) is for the $n = 1$ mode. It will turn out that this equation is sufficient to fix all the Λ's to any order and we will be able to determine the operator $\hat{O}$ completely just from this relation. Relations for higher modes that follow from equating the coefficients of the operators α's in equations (111) and (151) will then become consistency conditions. Moreover due to the cyclic symmetry in the string indices, we can set $r = 1$ without loss of generality. Thus setting $n = 1$ and $r = 1$ in both equations (151) and (111), we find

$$\begin{aligned} -\sum_{s=1}^{3}\sum_{m=0}^{\infty} G_{1m}^{1s}\alpha_m^s = \Bigg[& \sum_{q=1}^{\infty}\sum_{k=0}^{q} C_{qk}^{[-1]} e^{k\Lambda_0}\alpha_{-1+q}^1 - \sum_{s=1}^{3}\sum_{m=0}^{\infty} M_{1m}^{1s} e^{(m+1)\Lambda_0} \\ & \times \left(\alpha_m^s + \sum_{q=1}^{\infty}\sum_{k=0}^{q} C_{qk}^{[m]} e^{k\Lambda_0}\alpha_{m+q}^s\right)\Bigg] \end{aligned} \tag{152}$$

If we compare the terms involving α_0^1 in (152), we find

$$-\sum_{s=1}^{3} G_{10}^{1s}\alpha_0^s = \sum_{k=0}^{1} C_{1k}^{[-1]} e^{k\Lambda_0}\alpha_0^1 - \sum_{s=1}^{3} M_{10}^{1s} e^{\Lambda_0}\alpha_0^s \tag{153}$$

It is important to notice here that α_0^1, α_0^2 and α_0^3 are not linearly independent. The redundancy can be removed using conservation of momentum. Expanding the sums in the above expression and then using the conservation of momentum; that is; $\alpha_0^1 + \alpha_0^2 + \alpha_0^3 = 0$ to eliminate α_0^3, we obtain

$$\begin{aligned} -\left[\left(G_{10}^{11} - G_{10}^{13}\right)\alpha_0^1 + \left(G_{10}^{12} - G_{10}^{13}\right)\alpha_0^2\right] = &\left[C_{10}^{[-1]} + C_{11}^{[-1]} e^{\Lambda_0} - e^{\Lambda_0}\left(M_{10}^{11} - M_{10}^{13}\right)\right]\alpha_0^1 \\ &- e^{\Lambda_0}\left(M_{10}^{12} - M_{10}^{13}\right)\alpha_0^2 \end{aligned} \tag{154}$$

Since α_0^1 and α_0^2 are linearly independent, it follows that

$$e^{\Lambda_0}\left(M_{10}^{11} - M_{10}^{13}\right) - \left(G_{10}^{11} - G_{10}^{13}\right) = C_{10}^{[-1]} + C_{11}^{[-1]} e^{\Lambda_0} \tag{155}$$

$$e^{\Lambda_0}\left(M_{10}^{12} - M_{10}^{13}\right) - \left(G_{10}^{12} - G_{10}^{13}\right) = 0 \tag{156}$$

The second equation gives the value of Λ_0 in terms of the coupling matrices G and M

$$\Lambda_0 = \ln\left(G_{10}^{12} - G_{10}^{13}\right) \tag{157}$$

In arriving at the above equation, we used the fact that $M_{10}^{13} = 0$ and $M_{10}^{12} = 1$. The numerical values of the matrix elements G_{10}^{12} and G_{10}^{13} are computed in appendix B. From appendix B, we have

$$-G_{2n+1\ 0}^{13} = G_{2n+1\ 0}^{12} = \frac{1}{\sqrt{3}}\frac{a_{2n+1}}{2n+1} \tag{158}$$

and so for $n = 0$, we have

$$-G_{1\ 0}^{13} = G_{1\ 0}^{12} = \frac{1}{\sqrt{3}}a_1 = \frac{2}{3\sqrt{3}} \tag{159}$$

Putting these values in (157), we obtain

$$\Lambda_0 = \ln\frac{2^2}{3\sqrt{3}} \tag{160}$$

which is precisely the result obtained in [24].

Equation (155) gives the value of $\overline{\Lambda}_1$ ($\equiv \Lambda_1/\Lambda_0$). To see this, we first need to find the values of the constants $C_{10}^{[-1]}$ and $C_{11}^{[-1]}$. From equation (142) it follows that $C_{11}^{[-1]} = -C_{10}^{[-1]}$ and from equation (141) we find that $C_{10}^{[-1]} = C_{00}^{[-1]}\overline{\Lambda}_1$ but $C_{00}^{[-1]} = 1$ by equation (142); and so equation (155) gives

$$-\left(G_{10}^{11} - G_{10}^{13}\right) = \overline{\Lambda}_1\left(1 - e^{\Lambda_0}\right) \tag{161}$$

where again the last step in obtaining the above result follows from the fact that $M^{11}_{10} = M^{13}_{10} = 0$. Solving the above equation for $\overline{\Lambda}_1$ and then using (157) to eliminate Λ_0, we find

$$\overline{\Lambda}_1 = -\frac{G^{11}_{10} - G^{13}_{10}}{1 - (G^{12}_{10} - G^{13}_{10})} \tag{162}$$

where we have used $M^{12}_{10} = 1$. Making use of the explicit values of the matrix elements G^{11}_{10} and G^{13}_{10} computed in appendix B

$$G^{11}_{2n+1\ 0} = 0, \qquad n = 0, 1, 3, ... \tag{163}$$

$$-G^{13}_{2n+1\ 0} = G^{12}_{2n+1\ 0} = \frac{1}{\sqrt{3}}\frac{a_{2n+1}}{2n+1}, \qquad n = 0, 1, 3, ..., \tag{164}$$

equation (162) becomes

$$\overline{\Lambda}_1 = -\frac{6}{11}\sqrt{3} - \frac{8}{11} \tag{165}$$

Once more, this is the same result obtained in [24] using the full string formulation of Witten's string theory of open bosonic strings. Thus at least at the first level, the comma theory gives the same physics as Witten's theory. We will see that this conclusion in fact holds at any level.

Now we are in the position to derive a recursion relations for the $\overline{\Lambda}'$s. Using the fact that $M^{11}_{nm} = M^{13}_{nm} = 0$, equation (152) becomes

$$-\sum_{s=1}^{3}\sum_{m=0}^{\infty} G^{1s}_{1m}\alpha^s_m = \left[\sum_{q=1}^{\infty}\sum_{k=0}^{q} C^{[-1]}_{qk} e^{k\Lambda_0}\alpha^1_{-1+q} - \sum_{m=0}^{\infty} M^{12}_{1m} e^{(m+1)\Lambda_0} \times \left(\alpha^2_m + \sum_{q=1}^{\infty}\sum_{k=0}^{q} C^{[m]}_{qk} e^{k\Lambda_0}\alpha^2_{m+q}\right)\right] \tag{166}$$

If we now compare the coefficients for α^1_m, we find

$$-G^{11}_{1m} = \sum_{k=0}^{m+1} C^{[-1]}_{m+1k} e^{k\Lambda_0} \tag{167}$$

valid for $m = 1, 2, 3, ...$ Using equations (142) to eliminate $C^{[-1]}_{m+1m+1}$, the above expression becomes

$$-G^{11}_{1m} = \sum_{k=0}^{m} C^{[-1]}_{m+1k} e^{k\Lambda_0} - \sum_{k=0}^{m} C^{[-1]}_{m+1k} e^{(m+1)\Lambda_0} \tag{168}$$

or alternatively

$$-G^{11}_{1m} = \left(1 - e^{(m+1)\Lambda_0}\right) C^{[-1]}_{m+1\ 0} + \sum_{k=1}^{m} C^{[-1]}_{m+1k}\left(e^{k\Lambda_0} - e^{(m+1)\Lambda_0}\right) \tag{169}$$

Setting $q = m+1$, $k = 0$, $r = 1$ and $n = -1$ in (141), we have

$$C^{[-1]}_{m+1\,0} = -\frac{1}{m+1}\sum_{l=0}^{m}(-1+l)\,C^{[-1]}_{l\,0}\overline{\Lambda}_{m+1-l} \tag{170}$$

Using $C^{[-1]}_{00} = 1$, and multiplying both sides by $(m+1)$, the above expression takes the form

$$(m+1)\,C^{[-1]}_{m+1\,0} = \overline{\Lambda}_{m+1} - \sum_{l=2}^{m}(-1+l)\,C^{[-1]}_{l\,0}\overline{\Lambda}_{m+1-l} \tag{171}$$

Combining equations (169) and (171), we find

$$\overline{\Lambda}_{m+1} = -\frac{(m+1)}{1-e^{(m+1)\Lambda_0}}\left[G^{11}_{1m} + \sum_{k=1}^{m} C^{[-1]}_{m+1k}\left(e^{k\Lambda_0} - e^{(m+1)\Lambda_0}\right)\right]$$
$$+\sum_{l=2}^{m}(-1+l)\,C^{[-1]}_{l\,0}\overline{\Lambda}_{m+1-l} \tag{172}$$

valid for $m = 1, 2, 3, \ldots.$ According to equations (141), the right hand side of equation (172) is a function of $\overline{\Lambda}_0$, $\overline{\Lambda}_1$, $\overline{\Lambda}_2$, ..., $\overline{\Lambda}_m$. Thus equation (172) with the explicit values of $\overline{\Lambda}_0$ and $\overline{\Lambda}_1$ obtained in (160) and (165) generate all values of $\overline{\Lambda}\,'$s. To illustrate the use of the recursion relation in (172), we now proceed to compute the first few constants in the expansion of the conformal operator. The values of G^{rr}_{nm} has been computed in appendix B. For $m = odd \neq 1$, Equation (198) gives

$$G^{11}_{1m} = -\frac{2}{3^2}\,(-)^{(m-1)/2}\left[\frac{b_m + 2a_m}{1+m} + \frac{b_m - 2a_m}{1-m}\right] \tag{173}$$

where we have used the explicit values $a_1 = 2/3$ and $b_1 = 4/3$. To Compute the value of $\overline{\Lambda}_2$, we need the explicit value of G^{11}_{11}. The value of G^{11}_{11} may be obtained at once by setting $m = 1$ in equation (210)

$$G^{11}_{11} = -\frac{2^3}{3^3} - \frac{1}{\pi}\frac{2}{3\sqrt{3}}\left[\widetilde{E}^b_1 - 2\widetilde{E}^a_1\right] \tag{174}$$

where once again we used the fact that $a_1 = 2/3$, $b_1 = 4/3$. Using the explicit values of $\widetilde{E}^a_1$ and $\widetilde{E}^b_1$, which are given by equations (**??**) and (**??**)

$$\tilde{E}{}^a_{n=1} = \pi\sqrt{\frac{1}{3}}\left(\ln\frac{3}{2} + \frac{1}{6}\right) \tag{175}$$

and

$$\tilde{E}{}^b_{n=1} = 2\pi\sqrt{\frac{1}{3}}\left(\ln\frac{3}{2} - \frac{1}{12}\right) \tag{176}$$

the above expression for G^{11}_{11} becomes

$$G^{11}_{11} = -\frac{5}{3^3} \tag{177}$$

Setting $m = 1$ in equation (172) and substituting the explicit value of G^{11}_{11} obtained above, we get

$$\overline{\Lambda}_2 = -\frac{2}{1-e^{2\Lambda_0}}\left[G^{11}_{11} + C^{[-1]}_{2\;1}\left(e^{\Lambda_0} - e^{2\Lambda_0}\right)\right] \tag{178}$$

From equation (141); it follows that $C^{[-1]}_{2\;1} = 0$ and so substituting the explicit values of G^{11}_{11} and Λ_0, the above expression becomes

$$\overline{\Lambda}_2 = \frac{10}{11} \tag{179}$$

This is precisely the value obtained in [24] and so the comma theory gives the same physics at the second level. For $m = 2$, equation (172) gives

$$\overline{\Lambda}_3 = -\frac{3}{1-e^{3\Lambda_0}}\left[G^{11}_{12} + C^{[-1]}_{31}\left(e^{\Lambda_0} - e^{3\Lambda_0}\right) + C^{[-1]}_{32}\left(e^{2\Lambda_0} - e^{3\Lambda_0}\right)\right] + C^{[-1]}_{20}\overline{\Lambda}_1 \tag{180}$$

The G^{rr}_{nm} vanish for $r = 1, 2, 3$, and $n+m = odd$; see equation (195). The explicit values of the coefficients $C^{[-1]}_{31}$, $C^{[-1]}_{32}$, $C^{[-1]}_{20}$ are given by equation (141). For $C^{[-1]}_{20}$, equation (141) gives

$$C^{[-1]}_{20} = \frac{1}{2}C^{[-1]}_{00}\overline{\Lambda}_2 = \frac{1}{2}\overline{\Lambda}_2 \tag{181}$$

where we used the fact that $C^{[n]}_{00} = 1$. For $C^{[-1]}_{31}$, equation (141) gives

$$C^{[-1]}_{31} = -\frac{1}{2}C^{[-1]}_{21}\overline{\Lambda}_1 = 0 \tag{182}$$

where the value of $C^{[-1]}_{21} = 0$ follows at once from equation (141). Likewise one sees that

$$C^{[-1]}_{32} = -C^{[-1]}_{22}\overline{\Lambda}_1 = \left[C^{[-1]}_{20} + C^{[-1]}_{21}\right]\overline{\Lambda}_1 = \frac{1}{2}\overline{\Lambda}_2\overline{\Lambda}_1 \tag{183}$$

Putting all these results in (180), yields

$$\overline{\Lambda}_3 = -\frac{3}{1-e^{3\Lambda_0}}\left[\frac{1}{2}\overline{\Lambda}_2\overline{\Lambda}_1\left(e^{2\Lambda_0} - e^{3\Lambda_0}\right)\right] + \frac{1}{2}\overline{\Lambda}_2\overline{\Lambda}_1 \tag{184}$$

Substituting the explicit values of Λ_0, $\overline{\Lambda}_1$ and $\overline{\Lambda}_2$ in the above expression, we get

$$\overline{\Lambda}_3 = -\frac{30}{1417}\sqrt{3} - \frac{2360}{15\,587} \tag{185}$$

which is the desired result. To compute $\overline{\Lambda}_4$, we set $m = 3$ in equation (172)

$$\overline{\Lambda}_4 = -\frac{4}{1-e^{4\Lambda_0}}\left[G^{11}_{13} + C^{[-1]}_{41}\left(e^{\Lambda_0} - e^{4\Lambda_0}\right) + C^{[-1]}_{42}\left(e^{2\Lambda_0} - e^{4\Lambda_0}\right) + C^{[-1]}_{43}\left(e^{3\Lambda_0} - e^{4\Lambda_0}\right)\right] + C^{[-1]}_{2\,0}\overline{\Lambda}_2 + 2C^{[-1]}_{3\,0}\overline{\Lambda}_1 \tag{186}$$

The value of G^{11}_{13} is computed in appendix B. From equation (198), we find

$$G^{11}_{13} = \frac{2^5}{3^6} \tag{187}$$

The coefficients $C^{[-1]}_{2\,0}$, $C^{[-1]}_{30}$, $C^{[-1]}_{41}$, $C^{[-1]}_{42}$, $C^{[-1]}_{43}$ may be obtained from table C1.

$$\begin{aligned}
C^{[r,-1]}_{20} &= \frac{1}{2}\overline{\Lambda}^r_2 = \frac{5}{11} \\
C^{[-1]}_{30} &= C^{[r,-1]}_{30} = \frac{1}{3}\overline{\Lambda}^r_3 - \frac{1}{3\cdot 2}\overline{\Lambda}^r_1\overline{\Lambda}^r_2 = \frac{12\,960}{171\,457}\sqrt{3} + \frac{10\,240}{171\,457} \\
C^{[-1]}_{41} &= 0 \\
C^{[-1]}_{42} &= \frac{1}{2^2}\overline{\Lambda}^r_2\overline{\Lambda}^r_2 - \frac{1}{2}\overline{\Lambda}^r_1\overline{\Lambda}^r_1\overline{\Lambda}^r_2 = -\frac{480}{1331}\sqrt{3} - \frac{585}{1331} \\
C^{[-1]}_{43} &= \frac{2}{3}\overline{\Lambda}^r_1\overline{\Lambda}^r_3 + \frac{2}{3}\overline{\Lambda}^r_1\overline{\Lambda}^r_1\overline{\Lambda}^r_2 = \frac{1030\,080}{1886\,027}\sqrt{3} + \frac{1806\,840}{1886\,027}
\end{aligned}$$

Substituting these values in equation (186) and using the explicit values of $\overline{\Lambda}_0$, $\overline{\Lambda}_1$, $\overline{\Lambda}_2$ and $\overline{\Lambda}_3$ obtained earlier, we find

$$\overline{\Lambda}_4 = -\frac{7680}{670\,241}\sqrt{3} - \frac{7546}{60\,931} \tag{188}$$

For $\overline{\Lambda}_5$, we set $m = 4$ in equation (172) to find

$$\overline{\Lambda}_5 = -\frac{5}{1-e^{5\Lambda_0}}\left[0 + C^{[-1]}_{51}\left(e^{\Lambda_0} - e^{5\Lambda_0}\right) + C^{[-1]}_{52}\left(e^{2\Lambda_0} - e^{5\Lambda_0}\right) + C^{[-1]}_{53}\left(e^{3\Lambda_0} - e^{5\Lambda_0}\right) + C^{[-1]}_{54}\left(e^{4\Lambda_0} - e^{5\Lambda_0}\right)\right] + C^{[-1]}_{2\,0}\overline{\Lambda}_3 + 2C^{[-1]}_{3\,0}\overline{\Lambda}_2 + 3C^{[-1]}_{40}\overline{\Lambda}_1 \tag{189}$$

where we have used the fact that $G^{11}_{14} = 0$ (see equation (195) in appendix B). Substituting the explicit values of $\overline{\Lambda}_0$, $\overline{\Lambda}_1$, $\overline{\Lambda}_2$ and $\overline{\Lambda}_3$ and the explicit values of $C^{[-1]}_{51}$, $C^{[-1]}_{52}$, $C^{[-1]}_{53}$, $C^{[-1]}_{54}$, $C^{[-1]}_{2\,0}$, $C^{[-1]}_{3\,0}$ and $C^{[-1]}_{40}$ obtained in appendix C, the above expression becomes

$$\overline{\Lambda}_5 = \frac{7867620\,200\,382}{98058698\,647\,481}\sqrt{3} - \frac{18\,702\,116\,671\,704}{98\,058\,698\,647\,481} \tag{190}$$

Continuing this way we can compute $\overline{\Lambda}_n$ to any desired value of n and so this procedure gives the conformal operator required for the transformation between the comma 3-Vertex and the *SCSV* 3-Vertex to all levels. The fact that this operator turns out to be the same operator connecting Witten's interacting three vertex and the $SCSV$ three vertex is a non trivial check on the equivalence of the comma theory [11, 12, 13, 14, 15] and Witten's theory of the open bosonic string [7]. In appendix C, we give the first few values of the coefficients $C_{qk}^{[r,n]}$ as calculated from equations (141) and (142). In appendix C, we also give the first few values of the coefficients $\overline{\Lambda}$'s as calculated from equation (172).

A The Half-String Coupling Coefficients G_{nm}^{rs}

In this appendix we give the G_{nm}^{rs} coefficients as computed from (102) and (105). Here for the sake of being brief, we shall skip many of the straightforward algebra and spell out some of the algebraic steps whenever they are not trivial and require some attention.

A.1 The off diagonal elements

Comparing equations (102) and (105), we find

$$G_{nm}^{rs} = \frac{H_{nm}^{rs}}{n+m} + \frac{H_{nm}^{r\ -s}}{n+m}, \quad r = s = 1,2,3 \tag{191}$$

where the explicit form of the H_{nm}^{rs} matrix has the form

$$H_{nm}^{1\pm1} = \frac{1}{3}\begin{Bmatrix} (-1)^n \left(A_n B_m \pm B_n A_m\right) & n+m = even \\ 0 & n+m = odd \end{Bmatrix} \tag{192}$$

The A's and B's are the Fourier coefficients of the expansion of the functions $\left(\frac{1+ie^{\zeta}}{1-ie^{\zeta}}\right)^{1/3}$ and $\left(\frac{1+ie^{\zeta}}{1-ie^{\zeta}}\right)^{2/3}$ respectively. They are related to the coefficients a's and the b's in the expression of $\left(\frac{1+x}{1-x}\right)^{1/3}$ and $\left(\frac{1+x}{1-x}\right)^{2/3}$ respectively. Namely, we have

$$\begin{aligned} A(B)_n &= (-)^{n/2}\, a\,(b)_n\,, \qquad n = even \\ A(A)_n &= (-)^{(n-1)/2}\, a\,(b)_n\,, \quad n = odd \end{aligned} \tag{193}$$

Let us first consider the special case of $r = s$. In this case, we have

$$G_{nm}^{11} = G_{nm}^{22} = G_{nm}^{33} = \frac{H_{nm}^{11}}{n+m} + \frac{H_{nm}^{1\ -1}}{n+m} \tag{194}$$

From (192) it follows that

$$G_{2n+1\ 2m}^{rr} = G_{2m\ 2n+1}^{rr} = 0, \quad n,m \geq 0,\ r = 1,2,3 \tag{195}$$

For $n + m = even$, we have

$$\begin{aligned} H_{2n2m}^{1\pm1} &= \frac{1}{3}\left(a_{2n}b_{2m} \pm b_{2n}a_{2m}\right) \\ H_{2n+12m+1}^{1\pm1} &= -\frac{1}{3}\left(a_{2n+1}b_{2m+1} \pm b_{2n+1}a_{2m+1}\right) \end{aligned} \tag{196}$$

and so for $n \neq m$, one obtains

$$G^{rr}_{2n2m} = \frac{1}{3}(-)^{n+m}\left[\frac{a_{2n}b_{2m} + b_{2n}a_{2m}}{2n+2m} + \frac{a_{2n}b_{2m} - b_{2n}a_{2m}}{2n-2m}\right] \tag{197}$$

and

$$G^{rr}_{2n+12m+1} = -\frac{1}{3}(-)^{n+m}\left[\frac{a_{2n+1}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n+1)+(2m+1)} + \frac{a_{2n+1}b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n+1)-(2m+1)}\right] \tag{198}$$

Both of equations (197) and (198) are valid for $n, m \geq 0$. For $n = 0$ and $m \neq 0$, equation (197) gives G^{rr}_{2n2m}. The case $m = n$; that is, the diagonal elements G^{rr}_{nn} can be obtained as a limiting process of equations (197) and (198). This we shall do next. First we consider $G^{rr}_{2n+12n+1}$. We define $G^{rr}_{2n+12n+1}$ as

$$\begin{aligned} G^{rr}_{2n+12n+1} &= \lim_{m\to n} G^{rr}_{2n+12m+1} \\ &= \lim_{m\to n}\left\{-\frac{1}{3}(-)^{n+m}\left[\frac{a_{2n+1}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n+1)+(2m+1)} + \frac{a_{2n+1}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n+1)-(2m+1)}\right]\right\} \\ &= -\frac{2a_{2n+1}b_{2n+1}}{3.2(2n+1)} - \frac{1}{3}\lim_{\varepsilon\to 0}(-)^{n+m}\frac{a_{2n+1}b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n+1)-(2m+1)} \end{aligned} \tag{199}$$

Letting $n - m = \varepsilon$ in the term involving the limit, the above expression reduces to

$$G^{rr}_{2n+12n+1} = -\frac{2a_{2n+1}b_{2n+1}}{3.2(2n+1)} - \frac{1}{3}\lim_{\varepsilon\to 0}(-)^{\varepsilon}\frac{a_{2n+1}b_{2n+1-2\varepsilon} - b_{2n+1}a_{2n+1-2\varepsilon}}{2\varepsilon} \tag{200}$$

Now let us evaluate the limit in the above equation

$$\lim_{\varepsilon\to 0}(-)^{\varepsilon}\frac{a_{2n+1}b_{2n+1-2\varepsilon} - b_{2n+1}a_{2n+1-2\varepsilon}}{2\varepsilon} \tag{201}$$

This we do by using the integral representation for the Taylor modes; that is

$$a_{2n+1} = \frac{1}{\pi}\,\mathrm{Si}\,n\left[\frac{\pi}{3}\right]\int_1^\infty \frac{dx}{x^{2n+2}}\left[\left(\frac{x+1}{x-1}\right)^{1/3} + \left(\frac{x+1}{x-1}\right)^{-1/3}\right] \tag{202}$$

and

$$b_{2n+1} = \frac{1}{\pi}\,\mathrm{Si}\,n\left[\frac{2\pi}{3}\right]\int_1^\infty \frac{dx}{x^{2n+2}}\left[\left(\frac{x+1}{x-1}\right)^{2/3} + \left(\frac{x+1}{x-1}\right)^{-2/3}\right] \tag{203}$$

where $n = 0, 1, 2, \ldots$. It follows now that $a_{2n+1-2\varepsilon}$ and $b_{2n+1-2\varepsilon}$ are given by replacing $2n+1$ by $2n+1-2\varepsilon$ in the above equations. Thus we have

$$a_{2n+1-2\varepsilon} = \frac{1}{\pi}\frac{\sqrt{3}}{2}\int_1^\infty \frac{dx}{x^{2n+2-2\varepsilon}}\left[\left(\frac{x+1}{x-1}\right)^{1/3} + \left(\frac{x+1}{x-1}\right)^{-1/3}\right] \tag{204}$$

and

$$b_{2n+1-2\varepsilon} = \frac{1}{\pi}\frac{\sqrt{3}}{2}\int_1^{\infty}\frac{dx}{x^{2n+2-2\epsilon}}\left[\left(\frac{x+1}{x-1}\right)^{2/3}+\left(\frac{x+1}{x-1}\right)^{-2/3}\right] \tag{205}$$

Substituting these equations into (201) and rearranging terms, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(-)^{\varepsilon}\,\frac{a_{2n+1}b_{2n+1-2\varepsilon}-b_{2n+1}a_{2n+1-2\varepsilon}}{2\varepsilon}\\
=\;&\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)^2\lim_{\varepsilon\to 0}(-)^{\varepsilon}\,\frac{1}{2\epsilon}\left\{\int_1^{\infty}\frac{dx}{x^{2n+2}}\int_1^{\infty}\frac{dy}{y^{2n+2}}\left(\frac{1}{y^{-2\epsilon}}-\frac{1}{x^{-2\epsilon}}\right)\right.\\
&\left.\left[\left(\frac{x+1}{x-1}\right)^{1/3}+\left(\frac{x+1}{x-1}\right)^{-1/3}\right]\left[\left(\frac{y+1}{y-1}\right)^{2/3}+\left(\frac{y+1}{y-1}\right)^{-2/3}\right]\right\}
\end{aligned}$$

Using the fact that

$$\lim_{\varepsilon\to 0}(-)^{\varepsilon}\,(1/2\epsilon)\left(1/y^{-2\epsilon}-1/x^{-2\epsilon}\right)=\ln y-\ln x \tag{206}$$

the above expression becomes

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(-)^{\varepsilon}\,\frac{a_{2n+1}b_{2n+1-2\varepsilon}-b_{2n+1}a_{2n+1-2\varepsilon}}{2\varepsilon}\\
=\;&\left\{\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)\int_1^{\infty}\frac{dx}{x^{2n+2}}\left[\left(\frac{x+1}{x-1}\right)^{1/3}+\left(\frac{x+1}{x-1}\right)^{-1/3}\right]\right.\\
&\left.\times\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)\int_1^{\infty}\frac{\ln y\,dy}{y^{2n+2}}\left[\left(\frac{y+1}{y-1}\right)^{2/3}+\left(\frac{y+1}{y-1}\right)^{-2/3}\right]\right\}\\
&-\left\{\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)\int_1^{\infty}\frac{dy}{y^{2n+2}}\left[\left(\frac{y+1}{y-1}\right)^{2/3}+\left(\frac{y+1}{y-1}\right)^{-2/3}\right]\right.\\
&\left.\times\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)\int_1^{\infty}\frac{\ln x\,dx}{x^{2n+2}}\left[\left(\frac{x+1}{x-1}\right)^{1/3}+\left(\frac{x+1}{x-1}\right)^{-1/3}\right]\right\}
\end{aligned}$$

Using equation (**??**), the above expression reduces to

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(-)^{\varepsilon}\,\frac{a_{2n+1}b_{2n+1-2\varepsilon}-b_{2n+1}a_{2n+1-2\varepsilon}}{2\varepsilon}\\
=\;&a_{2n+1}\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)\int_1^{\infty}\frac{\ln y\,dy}{y^{2n+2}}\left[\left(\frac{y+1}{y-1}\right)^{2/3}+\left(\frac{y+1}{y-1}\right)^{-2/3}\right]\\
&-b_{2n+1}\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)\int_1^{\infty}\frac{\ln x\,dx}{x^{2n+2}}\left[\left(\frac{x+1}{x-1}\right)^{1/3}+\left(\frac{x+1}{x-1}\right)^{-1/3}\right]
\end{aligned} \tag{207}$$

Making use of the following identities derived in appendix B

$$\sum_{k=0}^{\infty}a_{2k}x^{2k}=\frac{1}{2}\left[\left(\frac{1+x}{1-x}\right)^{1/3}+\left(\frac{1+x}{1-x}\right)^{-1/3}\right]$$

and

$$\sum_{k=0}^{\infty} b_{2k} x^{2k} = \frac{1}{2}\left[\left(\frac{x+1}{x-1}\right)^{2/3} + \left(\frac{x+1}{x-1}\right)^{-2/3}\right]$$

equation (207) can be written in the form

$$\lim_{\varepsilon\to 0} (-)^{\varepsilon} \frac{a_{2n+1}b_{2n+1-2\varepsilon} - b_{2n+1}a_{2n+1-2\varepsilon}}{2\varepsilon} = \left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)$$
$$\times \left[a_{2n+1}\int_1^{\infty} \frac{\ln y dy}{y^{2n+2}} 2\sum_{k=0}^{\infty} b_{2k} y^{-2k} - b_{2n+1}\int_1^{\infty} \frac{\ln x dx}{x^{2n+2}} 2\sum_{k=0}^{\infty} a_{2k} x^{-2k}\right]$$

Since the sums in the above equation are uniformly convergent, we can integrate term by term. Thus term by term integration leads to

$$\lim_{\varepsilon\to 0} (-)^{\varepsilon} \frac{a_{2n+1}b_{2n+1-2\varepsilon} - b_{2n+1}a_{2n+1-2\varepsilon}}{2\varepsilon}$$
$$= \left(\frac{\sqrt{3}}{\pi}\right)\left[a_{2n+1}\sum_{k=0}^{\infty} b_{2k}\int_1^{\infty} \frac{\ln y dy}{y^{2n+2k+2}} - b_{2n+1}\sum_{k=0}^{\infty} a_{2k}\int_1^{\infty} \frac{\ln x dx}{x^{2n+2k+2}}\right] \tag{208}$$

The improper integral appearing in the above expression can be easily done by partial integration

$$\int_1^{\infty} \frac{\ln x dx}{x^{2n+2k+2}} = \frac{1}{(2n+2k+1)^2}$$

and so equation (208) becomes

$$\lim_{\varepsilon\to 0} (-)^{\varepsilon} \frac{a_{2n+1}b_{2n+1-2\varepsilon} - b_{2n+1}a_{2n+1-2\varepsilon}}{2\varepsilon}$$
$$= \left(\frac{\sqrt{3}}{\pi}\right)\left[a_{2n+1}\sum_{k=0}^{\infty} \frac{b_{2k}}{(2n+2k+1)^2} - b_{2n+1}\sum_{k=0}^{\infty} \frac{a_{2k}}{(2n+2k+1)^2}\right]$$
$$= \frac{\sqrt{3}}{\pi}\left(a_{2n+1}\widetilde{E}^b_{2n+1} - b_{2n+1}\widetilde{E}^a_{2n+1}\right)$$

Thus we have

$$\lim_{m\to n} (-)^{n+m} \frac{a_{2n+1}b_{2m+1} - b_{2n+1}a_{2n+1}}{(2n+1)-(2m+1)} = \lim_{\varepsilon\to 0} (-)^{\epsilon} \frac{a_{2n+1}b_{2m+1-2\epsilon} - b_{2n+1}a_{2n+1-2\epsilon}}{2\varepsilon}$$
$$= \frac{\sqrt{3}}{\pi}\left(a_{2n+1}\widetilde{E}^b_{2n+1} - b_{2n+1}\widetilde{E}^a_{2n+1}\right) \tag{209}$$

Using the above identity, equation (200) yields

$$G^{rr}_{2n+12n+1} = -\frac{2a_{2n+1}b_{2n+1}}{3.2(2n+1)} - \frac{1}{\pi}\sqrt{\frac{1}{3}}\left[a_{2n+1}\widetilde{E}^b_{2n+1} - b_{2n+1}\widetilde{E}^a_{2n+1}\right] \tag{210}$$

Likewise, G^{rr}_{2n2n} may be obtained by a limiting procedure. We define G^{rr}_{2n2n} as the limit of G^{rr}_{2n2m} as $m \to n$

$$G^{rr}_{2n2n} = \lim_{m\to n} G^{rr}_{2n2m} = \frac{2a_{2n}b_{2n}}{3.2(2n)} + \frac{1}{3}\lim_{m\to n}(-)^{n+m}\,\frac{a_{2n}b_{2m} - b_{2n}a_{2m}}{2n-2m} \tag{211}$$

Letting $n - m = \epsilon$ in the term involving the limit, the above expression reduces to

$$G^{rr}_{2n2n} = \frac{2a_{2n}b_{2n}}{3.2(2n)} + \frac{1}{3}\lim_{\epsilon\to 0}(-)^{\epsilon}\,\frac{a_{2n}b_{2n-2\epsilon} - b_{2n}a_{2n-2\epsilon}}{2\epsilon} \tag{212}$$

We proceed to evaluate

$$\lim_{\epsilon\to 0}(-)^{\epsilon}\,\frac{a_{2n}b_{2n-2\epsilon} - b_{2n}a_{2n-2\epsilon}}{2\epsilon} \tag{213}$$

This we do by using the integral representation for the Taylor modes

$$a_{2n} = \frac{1}{\pi}\frac{\sqrt{3}}{2}\int_1^\infty \frac{dx}{x^{2n+1}}\left[\left(\frac{x+1}{x-1}\right)^{1/3} - \left(\frac{x+1}{x-1}\right)^{-1/3}\right] \tag{214}$$

and

$$b_{2n} = \frac{1}{\pi}\frac{\sqrt{3}}{2}\int_1^\infty \frac{dx}{x^{2n+1}}\left[\left(\frac{x+1}{x-1}\right)^{2/3} - \left(\frac{x+1}{x-1}\right)^{-2/3}\right] \tag{215}$$

valid for $n > 0$. For $a_{2n-2\varepsilon}$ and $b_{2n-2\varepsilon}$ we replace $2n$ by $2n - 2\varepsilon$ in the above equations

$$a_{2n-2\varepsilon} = \frac{1}{\pi}\frac{\sqrt{3}}{2}\int_1^\infty \frac{dx}{x^{2n+1-2\varepsilon}}\left[\left(\frac{x+1}{x-1}\right)^{1/3} - \left(\frac{x+1}{x-1}\right)^{-1/3}\right]$$

$$b_{2n-2\varepsilon} = \frac{1}{\pi}\frac{\sqrt{3}}{2}\int_1^\infty \frac{dx}{x^{2n+1-2\epsilon}}\left[\left(\frac{x+1}{x-1}\right)^{2/3} - \left(\frac{x+1}{x-1}\right)^{-2/3}\right]$$

Substituting into (213), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(-)^{\varepsilon}\,\frac{a_{2n}b_{2n-2\varepsilon} - b_{2n}a_{2n-2\varepsilon}}{2\varepsilon} \\
= &\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)^2 \lim_{\varepsilon\to 0}(-)^{\varepsilon}\,\frac{1}{2\epsilon}\left\{\int_1^\infty \frac{dx}{x^{2n+1}}\int_1^\infty \frac{dy}{y^{2n+1}}\left(\frac{1}{y^{-2\epsilon}} - \frac{1}{x^{-2\epsilon}}\right)\right. \\
&\left.\times\left[\left(\frac{x+1}{x-1}\right)^{1/3} - \left(\frac{x+1}{x-1}\right)^{-1/3}\right]\left[\left(\frac{y+1}{y-1}\right)^{2/3} - \left(\frac{y+1}{y-1}\right)^{-2/3}\right]\right\}
\end{aligned}$$

Making use of (206), the above expression becomes

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(-)^{\varepsilon}\,\frac{a_{2n}b_{2n-2\varepsilon}-b_{2n}a_{2n-2\varepsilon}}{2\varepsilon} \\
=\;& a_{2n}\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)\int_1^\infty \frac{\ln y dy}{y^{2n+1}}\left[\left(\frac{y+1}{y-1}\right)^{2/3}-\left(\frac{y+1}{y-1}\right)^{-2/3}\right] \\
& -b_{2n}\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)\int_1^\infty \frac{\ln x dx}{x^{2n+1}}\left[\left(\frac{x+1}{x-1}\right)^{1/3}-\left(\frac{x+1}{x-1}\right)^{-1/3}\right] \qquad (216)
\end{aligned}$$

where we have used the fact that

$$a_{2n}=\frac{1}{\pi}\frac{\sqrt{3}}{2}\int_1^\infty \frac{dx}{x^{2n+1}}\left[\left(\frac{x+1}{x-1}\right)^{1/3}-\left(\frac{x+1}{x-1}\right)^{-1/3}\right]$$

and

$$b_{2n}=\frac{1}{\pi}\frac{\sqrt{3}}{2}\int_1^\infty \frac{dy}{y^{2n+1}}\left[\left(\frac{y+1}{y-1}\right)^{2/3}-\left(\frac{y+1}{y-1}\right)^{-2/3}\right]$$

From appendix B we have

$$\sum_{k=0}^{\infty} u_{2k+1}^{q/p}x^{-2k+1}=\frac{1}{2}\left[\left(\frac{x+1}{x-1}\right)^{q/p}-\left(\frac{x+1}{x-1}\right)^{-q/p}\right]$$

and so equation (216) can be written as

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(-)^{\varepsilon}\,\frac{a_{2n}b_{2n-2\varepsilon}-b_{2n}a_{2n-2\varepsilon}}{2\varepsilon} \\
=\;& a_{2n}\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)2\sum_{k=0}^{\infty}b_{2k+1}\int_1^\infty \frac{\ln y dy}{y^{2n+2k+2}}-b_{2n}\left(\frac{1}{\pi}\frac{\sqrt{3}}{2}\right)2\sum_{k=0}^{\infty}a_{2k+1}\int_1^\infty \frac{\ln x dx}{x^{2n+2k+2}}
\end{aligned}$$

In arriving at the above expression we have exchanged the summation and integration signs, an operation which is valid since the sums are uniformly convergent. Partial integration now yields

$$\lim_{\varepsilon\to 0}(-)^{\varepsilon}\,\frac{a_{2n}b_{2n-2\varepsilon}-b_{2n}a_{2n-2\varepsilon}}{2\varepsilon}=\frac{\sqrt{3}}{\pi}\left(a_{2n}\widetilde{O}_{2n}^{b}-b_{2n}\widetilde{O}_{2n}^{a}\right)$$

where

$$\widetilde{O}_{2n}^{a}=\sum_{k=0}^{\infty}\frac{a_{2k+1}}{(2n+2k+1)^2},\quad \widetilde{O}_{2n}^{b}=\sum_{k=0}^{\infty}\frac{b_{2k+1}}{(2n+2k+1)^2}$$

have been evaluated before (see appendix A).Thus we have

$$\lim_{m\to n}(-)^{n+m}\,\frac{a_{2n}b_{2m}-b_{2n}a_{2m}}{2n-2m}=\lim_{\varepsilon\to 0}(-)^{\varepsilon}\,\frac{a_{2n}b_{2n-2\varepsilon}-b_{2n}a_{2n-2\varepsilon}}{2\epsilon}=\frac{\sqrt{3}}{\pi}\left(a_{2n}\widetilde{O}_{2n}^{b}-b_{2n}\widetilde{O}_{2n}^{a}\right) \qquad (217)$$

Using the above identity, equation (217) yields

$$G^{rr}_{2n2n} = \frac{2a_{2n}b_{2n}}{3.2(2n)} + \frac{1}{\pi}\sqrt{\frac{1}{3}}\left(a_{2n}\widetilde{O}^{b}_{2n} - b_{2n}\widetilde{O}^{a}_{2n}\right) \tag{218}$$

where $n = 1, 2, 3, ...$ For the diagonal elements in the rs indices to be complete we still need to evaluate G^{rr}_{00} .

Next we consider the rs elements for $r \neq s$. Once more comparing equations (102) and (105), we find

$$G^{rs}_{nn} = G^{rs}_{nm} + G^{r,-s}_{nm} \tag{219}$$

where

$$G^{r,\pm r}_{nm} = \frac{(-)^n}{3\,(n \pm m)}\begin{Bmatrix} A_nB_m \pm B_nA_m\ , & n+m = even \\ 0\ , & n+m = odd \end{Bmatrix}, \tag{220}$$

$$G^{r,\pm(r+1)}_{nm} = \frac{1}{6\,(n \pm m)}\begin{Bmatrix} -(-)^n\,(A_nB_m \pm B_nA_m)\ , & n+m = even \\ \sqrt{3}\,(A_nB_m \mp B_nA_m)\ , & n+m = odd \end{Bmatrix}, \tag{221}$$

and

$$G^{r,\pm(r-1)}_{nm} = \frac{1}{6\,(n \mp m)}\begin{Bmatrix} (-)^n\,(A_nB_m \mp B_nA_m)\ , & n+m = even \\ \sqrt{3}\,(A_nB_m \pm B_nA_m)\ , & n+m = odd \end{Bmatrix} \tag{222}$$

where $r + 1 = 4 \equiv 1$ and $r - 1 = 0 \equiv 3$. Now the G^{12}_{nm} coefficients

$$G^{12}_{nm} = G^{1,2}_{nm} + G^{1,-2}_{nm} \tag{223}$$

reads

$$G^{12}_{nm} = \frac{1}{6\,(n+m)}\begin{Bmatrix} -(-)^n\,(A_nB_m + B_nA_m) \\ \sqrt{3}\,(A_nB_m - B_nA_m) \end{Bmatrix} + \frac{1}{6\,(n-m)}\begin{Bmatrix} -(-)^n\,(A_nB_m - B_nA_m) \\ \sqrt{3}\,(A_nB_m + B_nA_m) \end{Bmatrix} \tag{224}$$

where the upper line is for $n + m = even$ and the lower line is for $n + m = odd$. Thus equation (224) gives

$$G^{12}_{2n2m} = -\frac{1}{6}\left[\frac{A_{2n}B_{2m} + B_{2n}A_{2m}}{(2n+2m)} + \frac{A_{2n}B_{2m} - B_{2n}A_{2m}}{(2n-2m)}\right] \tag{225}$$

or in terms of the a_n and b_n., the above expression reads

$$G^{12}_{2n2m} = -\frac{1}{6}\,(-)^{n+m}\left[\frac{a_{2n}b_{2m} + b_{2n}a_{2m}}{(2n+2m)} + \frac{a_{2n}b_{2m} - b_{2n}a_{2m}}{(2n-2m)}\right] \tag{226}$$

Likewise one finds

$$G^{12}_{2n+12m+1} = \frac{1}{6}\,(-)^{n+m}\left[\frac{a_{2n+1}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n+1)+(2m+1)} + \frac{a_{2n+1}b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n+1)-(2m+1)}\right] \tag{227}$$

For the off diagonal elements, we have $n + m = odd$ and equation (224) yields

$$G^{12}_{2n2m+1} = \frac{1}{2\sqrt{3}}\left[\frac{A_{2n}B_{2m+1} - B_{2n}A_{2m+1}}{(2n)+(2m+1)} + \frac{A_{2n}B_{2m+1} + B_{2n}A_{2m+1}}{(2n)-(2m+1)}\right] \tag{228}$$

or in terms of a_n and b_n., the above expression reads

$$G^{12}_{2n2m+1} = \frac{1}{2\sqrt{3}}(-)^{n+m}\left[\frac{a_{2n}b_{2m+1} - b_{2n}a_{2m+1}}{(2n)+(2m+1)} + \frac{a_{2n}b_{2m+1} + b_{2n}a_{2m+1}}{(2n)-(2m+1)}\right] \tag{229}$$

Likewise, we find

$$G^{12}_{2n+12m} = \frac{1}{2\sqrt{3}}(-)^{n+m}\left[\frac{a_{2n+1}b_{2m} - b_{2n+1}a_{2m}}{(2n+1)+(2m)} + \frac{a_{2n+1}b_{2m} + b_{2n+1}a_{2m}}{(2n+1)-(2m)}\right] \tag{230}$$

Next we consider the G^{23}_{nm} coefficients

$$G^{23}_{nm} = G^{2,3}_{nm} + G^{2,-3}_{nm} \tag{231}$$

Using (221), the above expression gives

$$G^{23}_{nm} = \frac{1}{6(n+m)}\begin{Bmatrix} -(-)^n(A_nB_m + B_nA_m) \\ \sqrt{3}(A_nB_m - B_nA_m) \end{Bmatrix} + \frac{1}{6(n-m)}\begin{Bmatrix} -(-)^n(A_nB_m - B_nA_m) \\ \sqrt{3}(A_nB_m + B_nA_m) \end{Bmatrix} \tag{232}$$

where the upper line is for $n+m = even$ and the lower line is for $n+m = odd$. Thus (232) gives

$$G^{23}_{2n2m} = -\frac{1}{6}\left[\frac{A_{2n}B_{2m} + B_{2n}A_{2m}}{(2n+2m)} + \frac{A_{2n}B_{2m} - B_{2n}A_{2m}}{(2n-2m)}\right] \tag{233}$$

or in terms of a_n and b_n., the above expression reads

$$G^{23}_{2n2m} = -\frac{1}{6}(-)^{n+m}\left[\frac{a_{2n}b_{2m} + b_{2n}a_{2m}}{(2n+2m)} + \frac{a_{2n}b_{2m} - b_{2n}a_{2m}}{(2n-2m)}\right] \tag{234}$$

which is equal to G^{12}_{2n2m} as expected (due to the cyclic symmetry in the string index). Similarly we have

$$G^{23}_{2n+12m+1} = \frac{1}{6}\left[\frac{A_{2n+1}B_{2m+1} + B_{2n+1}A_{2m+1}}{(2n+1)+(2m+1)} + \frac{A_{2n+1}B_{2m+1} - B_{2n+1}A_{2m+1}}{(2n-1)-(2m+1)}\right] \tag{235}$$

or in terms of a_n and b_n., the above expression reads

$$G^{23}_{2n+12m+1} = \frac{1}{6}(-)^{n+m}\left[\frac{a_{2n+1}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n+1)+(2m+1)} + \frac{a_{2n+1}b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n-1)-(2m+1)}\right] \tag{236}$$

which is equal to $G^{12}_{2n+12m+1}$ as expected. The off diagonal elements are given by (232) for $n+m = odd$. Thus we have,

$$G^{23}_{2n2m+1} = \frac{1}{2\sqrt{3}}\left[\frac{A_{2n}B_{2m+1} - B_{2n}A_{2m+1}}{(2n)+(2m+1)} + \frac{A_{2n}B_{2m+1} + B_{2n+1}A_{2m+1}}{(2n)-(2m+1)}\right] \tag{237}$$

or in terms of a_n and b_n., the above expression reads

$$G^{23}_{2n2m+1} = \frac{1}{2\sqrt{3}}(-)^{n+m}\left[\frac{a_{2n}b_{2m+1} - b_{2n}a_{2m+1}}{(2n)+(2m+1)} + \frac{a_{2n}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n)-(2m+1)}\right] \tag{238}$$

Once more this result is the same as G^{12}_{2n2m+1}. Likewise, we find

$$G^{23}_{2n+12m} = \frac{1}{2\sqrt{3}}(-)^{n+m}\left[\frac{a_{2n+1}b_{2m} - b_{2n+1}a_{2m}}{(2n+1)+(2m)} + \frac{a_{2n+1}b_{2m} + b_{2n+1}a_{2m}}{(2n+1)-(2m)}\right] \tag{239}$$

This is the exact result as G^{12}_{2n+12m}. Next we consider

$$G^{13}_{nm} = G^{1,3}_{nm} + G^{1,-3}_{nm} \tag{240}$$

Substituting (222) into the right hand side we obtain

$$G^{13}_{nm} = -\frac{1}{6(n-m)}\left\{\begin{matrix}(-)^n(A_nB_m - B_nA_m)\\ \sqrt{3}(A_nB_m + B_nA_m)\end{matrix}\right\} - \frac{1}{6(n+m)}\left\{\begin{matrix}(-)^n(A_nB_m + B_nA_m)\\ \sqrt{3}(A_nB_m - B_nA_m)\end{matrix}\right\} \tag{241}$$

where the upper line is for $n+m = even$ and the lower line is for $n+m = odd$. Thus (241) gives

$$G^{13}_{2n2m} = -\frac{1}{6}\left[\frac{A_{2n}B_{2m} - B_{2n}A_{2m}}{(2n-2m)} + \frac{A_{2n}B_{2m} + B_{2n}A_{2m}}{(2n+2m)}\right] \tag{242}$$

or in terms of a_n and b_n., the above expression reads

$$G^{13}_{2n2m} = -\frac{1}{6}(-)^{n+m}\left[\frac{a_{2n}b_{2m} - b_{2n}a_{2m}}{(2n-2m)} + \frac{a_{2n}b_{2m} + b_{2n}a_{2m}}{(2n+2m)}\right] \tag{243}$$

Observe that G^{13}_{2n2m} is the same as G^{12}_{2n2m}. Similarly for $G^{13}_{2n+12m+1}$ one obtains

$$G^{13}_{2n+12m+1} = \frac{1}{6}(-)^{n+m}\left[\frac{a_{2n+1}b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n+1)-(2m+1)} + \frac{a_{2n+1}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n-1)+(2m+1)}\right] \tag{244}$$

which is the same as $G^{12}_{2n+12m+1}$. The off diagonal elements are given by (241) for $n+m = odd$. Thus we have,

$$G^{13}_{2n2m+1} = -\frac{1}{2\sqrt{3}}\left[\frac{A_{2n}B_{2m+1} + B_{2n}A_{2m+1}}{(2n)-(2m+1)} + \frac{A_{2n}B_{2m+1} - B_{2n+1}A_{2m+1}}{(2n)+(2m+1)}\right] \tag{245}$$

or in terms of a_n and b_n, the above expression reads

$$G^{13}_{2n2m+1} = -\frac{1}{2\sqrt{3}}(-)^{n+m}\left[\frac{a_{2n}b_{2m+1} + b_{2n}a_{2m+1}}{(2n)-(2m+1)} + \frac{a_{2n}b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n)+(2m+1)}\right] \tag{246}$$

Observe that $G^{13}_{2n2m+1} = -G^{12}_{2n2m+1}$. Likewise for G^{13}_{2n+12m} one obtains

$$G^{13}_{2n+12m} = -\frac{1}{2\sqrt{3}}(-)^{n+m}\left[\frac{a_{2n+1}b_{2m} + b_{2n+1}a_{2m}}{(2n+1)-(2m)} + \frac{a_{2n+1}b_{2m} - b_{2n+1}a_{2m}}{(2n+1)+(2m)}\right] \tag{247}$$

Observe that $G^{13}_{2n+12m} = -G^{12}_{2n+12m}$. Now we consider

$$G^{21}_{nm} = G^{2,1}_{nm} + G^{2,-1}_{nm} \tag{248}$$

Making use of (222), the above expression becomes

$$G^{21}_{nm} = -\frac{1}{6\,(n-m)}\left\{\begin{matrix} (-)^n\,(A_nB_m - B_nA_m) \\ \sqrt{3}\,(A_nB_m + B_nA_m) \end{matrix}\right\} - \frac{1}{6\,(n+m)}\left\{\begin{matrix} (-)^n\,(A_nB_m + B_nA_m) \\ \sqrt{3}\,(A_nB_m - B_nA_m) \end{matrix}\right\} \tag{249}$$

where the upper line is for $n+m = \mathit{even}$ and the lower line is for $n+m = \mathit{odd}$. Thus (249) gives

$$G^{21}_{2n2m} = -\frac{1}{6}\left[\frac{A_{2n}B_{2m} - B_{2n}A_{2m}}{(2n-2m)} + \frac{A_{2n}B_{2m} + B_{2n}A_{2m}}{(2n+2m)}\right] \tag{250}$$

or in terms of a_n and b_n., the above expression reads

$$G^{13}_{2n2m} = -\frac{1}{6}\,(-)^{n+m}\left[\frac{a_{2n}b_{2m} - b_{2n}a_{2m}}{(2n-2m)} + \frac{a_{2n}b_{2m} + b_{2n}a_{2m}}{(2n+2m)}\right] \tag{251}$$

Again this result is the same as G^{12}_{2n2m}. Likewise for $G^{21}_{2n+12m+1}$ one obtains

$$G^{13}_{2n+12m+1} = \frac{1}{6}\,(-)^{n+m}\left[\frac{a_{2n+1}b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n+1)-(2m+1)} + \frac{a_{2n+1}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n+1)+(2m+1)}\right] \tag{252}$$

We note that this result is the same as $G^{12}_{2n+12m+1}$. For $G^{21}_{n+m=odd}$, the off diagonal elements are given by setting $n+m = \mathit{odd}$ in (241). Thus we have,

$$G^{21}_{2n2m+1} = -\frac{1}{2\sqrt{3}}\left[\frac{A_{2n}B_{2m+1} + B_{2n}A_{2m+1}}{(2n)-(2m+1)} + \frac{A_{2n}B_{2m+1} - B_{2n+1}A_{2m+1}}{(2n)+(2m+1)}\right] \tag{253}$$

or in terms of a_n and b_n., the above expression reads

$$G^{21}_{2n2m+1} = -\frac{1}{2\sqrt{3}}\,(-)^{n+m}\left[\frac{a_{2n}b_{2m+1} + b_{2n}a_{2m+1}}{(2n)-(2m+1)} + \frac{a_{2n}b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n)+(2m+1)}\right] \tag{254}$$

Observe that $G^{21}_{2n2m+1} = -G^{12}_{2n2m+1}$. Similarly for G^{21}_{2n+12m} we have

$$G^{21}_{2n+12m} = -\frac{1}{2\sqrt{3}}\,(-)^{n+m}\left[\frac{a_{2n+1}b_{2m} + b_{2n+1}a_{2m}}{(2n+1)-(2m)} + \frac{a_{2n+1}b_{2m} - b_{2n+1}a_{2m}}{(2n+1)+(2m)}\right] \tag{255}$$

Next we consider G^{31}_{nm}. In this case equation (219) gives

$$G^{31}_{nm} = G^{3,1}_{nm} + G^{3,-1} \tag{256}$$

Substituting equation (221) into the right hand side, we find

$$G^{31}_{nm} = -\frac{1}{6\,(n+m)}\left\{\begin{matrix} (-)^n\,(A_nB_m + B_nA_m) \\ \sqrt{3}\,(A_nB_m - B_nA_m) \end{matrix}\right\} + \frac{1}{6\,(n+m)}\left\{\begin{matrix} (-)^n\,(A_nB_m - B_nA_m) \\ \sqrt{3}\,(A_nB_m + B_nA_m) \end{matrix}\right\} \tag{257}$$

where the upper line is for $n+m = even$ and the lower line is for $n+m = odd$. Thus (257) gives

$$G^{31}_{2n2m} = -\frac{1}{6}\left[\frac{A_{2n}B_{2m} + B_{2n}A_{2m}}{(2n+2m)} + \frac{A_{2n}B_{2m} - B_{2n}A_{2m}}{(2n-2m)}\right] \tag{258}$$

or in terms of a_n and b_n., the above expression reads

$$G^{31}_{2n2m} = -\frac{1}{6}\,(-)^{n+m}\left[\frac{a_{2n}b_{2m} + b_{2n}a_{2m}}{(2n+2m)} + \frac{a_{2n}b_{2m} - b_{2n}a_{2m}}{(2n-2m)}\right] \tag{259}$$

where $n \neq m$. Likewise for $G^{31}_{2n+12m+1}$ we obtain

$$G^{31}_{2n+12m+1} = \frac{1}{6}\,(-)^{n+m}\left[\frac{a_{2n+1}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n+1)+(2m+1)} + \frac{a_{2n+1}b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n+1)-(2m+1)}\right] \tag{260}$$

For $G^{31}_{n+m=odd}$ equation (224) gives

$$G^{31}_{2n2m+1} = \frac{1}{2\sqrt{3}}\,(-)^{n+m}\left[\frac{a_{2n}b_{2m+1} - b_{2n}a_{2m+1}}{(2n)+(2m+1)} + \frac{a_{2n}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n)-(2m+1)}\right] \tag{261}$$

and

$$G^{31}_{2n+12m} = \frac{1}{2\sqrt{3}}\,(-)^{n+m}\left[\frac{a_{2n+1}b_{2m} - b_{2n+1}a_{2m}}{(2n+1)+(2m)} + \frac{a_{2n+1}b_{2m} + b_{2n+1}a_{2m}}{(2n+1)-(2m)}\right] \tag{262}$$

Observe that $G^{31}_{nm} = G^{12}_{nm}$ for both $n+m = even$ and $n+m = odd$ as expected from the cyclic symmetry in the string index. Next we consider G^{32}_{nm}, which follows from (219). Thus we have

$$G^{32}_{nm} = G^{3,2}_{nm} + G^{3,-2}_{nm} \tag{263}$$

Using equation (222) the above expression becomes

$$G^{32}_{nm} = -\frac{1}{6\,(n-m)}\left\{\begin{matrix}(-)^n\,(A_nB_m - B_nA_m)\\ \sqrt{3}\,(A_nB_m + B_nA_m)\end{matrix}\right\} - \frac{1}{6\,(n+m)}\left\{\begin{matrix}(-)^n\,(A_nB_m + B_nA_m)\\ \sqrt{3}\,(A_nB_m - B_nA_m)\end{matrix}\right\} \tag{264}$$

where the upper line is for $n+m = even$ and the lower line is for $n+m = odd$. Thus for $n = even$ and $m = even$, (264) gives

$$G^{32}_{2n2m} = -\frac{1}{6}\left[\frac{A_{2n}B_{2m} - B_{2n}A_{2m}}{(2n-2m)} + \frac{A_{2n}B_{2m} + B_{2n}A_{2m}}{(2n+2m)}\right] \tag{265}$$

or in terms of a_n and b_n., the above expression reads

$$G^{32}_{2n2m} = -\frac{1}{6}\,(-)^{n+m}\left[\frac{a_{2n}b_{2m} - b_{2n}a_{2m}}{(2n-2m)} + \frac{a_{2n}b_{2m} + b_{2n}a_{2m}}{(2n+2m)}\right] \tag{266}$$

which is the same as G^{12}_{2n2m}. Likewise for $n = odd$ and $m = odd$, we find

$$G^{32}_{2n+12m+1} = \frac{1}{6}(-)^{n+m}\left[\frac{a_{2n+1}b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n+1)-(2m+1)} + \frac{a_{2n+1}b_{2m+1} + b_{2n+1}a_{2m+1}}{(2n+1)+(2m+1)}\right] \tag{267}$$

This is the same result we obtained for $G^{12}_{2n+12m+1}$. For the off diagonal elements with $n = even$ and $m = odd$, equation (241) gives

$$G^{32}_{2n2m+1} = -\frac{1}{2\sqrt{3}}(-)^{n+m}\left[\frac{a_{2n}b_{2m+1} + b_{2n}a_{2m+1}}{(2n)-(2m+1)} + \frac{a_{2n}b_{2m+1} - b_{2n}a_{2m+1}}{(2n)+(2m+1)}\right] \tag{268}$$

Observe that $G^{32}_{2n2m+1} = G^{21}_{2n2m+1} = -G^{23}_{2n2m+1}$ as expected. Similarly for for $n = odd$ and $m = even$, we find

$$G^{32}_{2n+12m} = -\frac{1}{2\sqrt{3}}(-)^{n+m}\left[\frac{a_{2n+1}b_{2m} + b_{2n+1}a_{2m}}{(2n+1)-(2m)} + \frac{a_{2n+1}b_{2m} - b_{2n+1}a_{2m}}{(2n+1)+(2m)}\right] \tag{269}$$

Once more we see that $G^{32}_{2n+12m} = G^{21}_{2n+12m} = -G^{23}_{2n+12m}$ as expected.

A.2 The diagonal elements

Next we need to consider the diagonal elements. We first consider the case when $m = n = even$. The diagonal elements are obtained by a limiting procedure of the of diagonal elements as we have seen before. Taking the limit of $m \to n$ in equation (224) we have

$$G^{12}_{2n2n} = \lim_{m\to n} G^{12}_{2n2m} = -\frac{1}{6}\lim_{m\to n}\left[\frac{a_{2n}b_{2m} + b_{2n}a_{2m}}{2n+2m} + \frac{a_{2n}b_{2m} - b_{2n}a_{2m}}{2n-2m}\right] \tag{270}$$

Let $n - m = \varepsilon$, then the above expression becomes

$$G^{12}_{2n2n} = -\frac{1}{6}\left(\frac{a_{2n}b_{2m}}{2n}\right) - \frac{1}{6}\lim_{\varepsilon\to 0}\frac{a_{2n}b_{2n-2\varepsilon} - b_{2n}a_{2n-2\varepsilon}}{2n} \tag{271}$$

Using the fact that (see previous calculations)

$$\lim_{\varepsilon\to 0}\frac{a_{2n}b_{2n-2\varepsilon} - b_{2n}a_{2n-2\varepsilon}}{2n} = \frac{\sqrt{3}}{\pi}\left(a_{2n}\widetilde{O}^{b}_{2n} - b_{2n}\widetilde{O}^{a}_{2n}\right) \tag{272}$$

Equation (271) yields

$$G^{12}_{2n2n} = -\frac{1}{6}\frac{a_{2n}b_{2n}}{2n} - \frac{1}{6}\frac{\sqrt{3}}{\pi}\left(a_{2n}\widetilde{O}^{b}_{2n} - b_{2n}\widetilde{O}^{a}_{2n}\right) \tag{273}$$

where the quantities $\widetilde{O}^{a}_{2n}$ and $\widetilde{O}^{b}_{2n}$ have been defined in appendix A. For the case $n = m = odd$, taking the limit of $m \to n$ in equation (227) we have

$$G^{12}_{2n+12n+1} = \lim_{m\to n} N^{12}_{2n+12m+1} = \frac{1}{6}\frac{a_{2n+1}b_{2n+1}}{2n+1} + \frac{1}{6}\lim_{m\to n}(-)^{n+m}\frac{a_{2n+1}+1b_{2m+1} - b_{2n+1}a_{2m+1}}{(2n+1)-(2m+1)} \tag{274}$$

If we now let $n - m = \varepsilon$, then the above expression becomes

$$G^{12}_{2n+12n+1} = \frac{1}{6}\left(\frac{a_{2n+1}b_{2n+1}}{2n+1}\right) + \frac{1}{6}\lim_{\varepsilon\to 0}(-)^{\varepsilon}\ \frac{a_{2n+1} + b_{2n+1-2\varepsilon} - b_{2n+1}a_{2n+1-2\varepsilon}}{2\varepsilon} \tag{275}$$

The limit in the above expression is given by equation (**??**) and so the above expression reduces to

$$G^{12}_{2n+12n+1} = \frac{1}{6}\frac{a_{2n+1}b_{2n+1}}{2n+1} + \frac{1}{6}\frac{\sqrt{3}}{\pi}\left(a_{2n+1}\widetilde{E}^{b}_{2n} - b_{2n+1}\widetilde{E}^{a}_{2n}\right) \tag{276}$$

From the cyclic symmetry of the strings, it follows that

$$G^{23}_{2n2n} = G^{31}_{2n2n} = G^{12}_{2n2n} = -\frac{1}{6}\frac{a_{2n}b_{2n}}{2n} - \frac{1}{6}\frac{\sqrt{3}}{\pi}\left(a_{2n}\widetilde{O}^{b}_{2n} - b_{2n}\widetilde{O}^{a}_{2n}\right) \tag{277}$$

and

$$G^{23}_{2n+12n+1} = G^{31}_{2n+12n+1} = G^{12}_{2n+12n+1} = \frac{1}{6}\left(\frac{a_{2n+1}b_{2n+1}}{2n+1}\right) + \frac{1}{6}\frac{\sqrt{3}}{\pi}\left(a_{2n+1}\widetilde{E}^{b}_{2n+1} - b_{2n+1}\widetilde{E}^{a}_{2n+1}\right) \tag{278}$$

In fact for the diagonal elements we have seen that $G_{even\ even}$ has the same expression for all $r \neq s$. Thus it follows that

$$\begin{aligned} G^{13}_{2n2n} &= G^{32}_{2n2n} = G^{21}_{2n2n} = G^{12}_{2n2n} = G^{23}_{2n2n} = G^{31}_{2n2n} \\ &= -\frac{1}{6}\frac{a_{2n}b_{2n}}{2n} - \frac{1}{6}\frac{\sqrt{3}}{\pi}\left(a_{2n}\widetilde{O}^{b}_{2n} - b_{2n}\widetilde{O}^{a}_{2n}\right) \end{aligned} \tag{279}$$

Likewise we have seen that $G_{odd\ odd}$ has the same expression for all $r \neq s$. Therefore we have

$$\begin{aligned} G^{13}_{2n+12n+1} &= G^{32}_{2n+12n+1} = G^{21}_{2n+12n+1} = G^{12}_{2n+12n+1} = G^{23}_{2n+12n+1} = G^{31}_{2n+12n+1} \\ &= \frac{1}{6}\frac{a_{2n+1}b_{2n+1}}{2n+1} + \frac{1}{6}\frac{\sqrt{3}}{\pi}\left(a_{2n+1}\widetilde{E}^{b}_{2n+1} - b_{2n+1}\widetilde{E}^{a}_{2n+1}\right) \end{aligned} \tag{280}$$

This completes the computation of all the G-coefficients.

B The Neumann coefficients $\widetilde{N}^{rs}_{nm}$ of reference [19]

The Neumann coefficients have been considered in [19]. In this section we summarize some of the results obtained in reference [19] and also we compute some others which are needed in the proof of the Ward-like identities considered in chapter 4 but have not been computed in reference [19]. In [19] the Neumann coefficients $\widetilde{N}^{rs}_{nm}$ are given by

$$\widetilde{N}^{rs}_{nm} = \widetilde{N}^{r,s}_{nm} + \widetilde{N}^{r,-s}_{nm} \tag{281}$$

where

$$\widetilde{N}^{r,,\pm s}_{nm} = -\frac{(-)^n}{3(n\pm m)}\begin{Bmatrix} B_nA_m \pm A_nB_m & n+m=even \\ 0 & n+m=odd \end{Bmatrix}, \qquad (282)$$

$$\widetilde{N}^{r,,\pm(r+1)}_{nm} = \frac{1}{6(n\pm m)}\begin{Bmatrix} (-)^n(B_nA_m \pm A_nB_m) & n+m=even \\ \sqrt{3}(B_nA_m \mp A_nB_m) & n+m=odd \end{Bmatrix} \qquad (283)$$

and

$$\widetilde{N}^{r,,\pm(r-1)}_{nm} = \frac{1}{6(n\mp m)}\begin{Bmatrix} (-)^n(B_nA_m \mp A_nB_m) & n+m=even \\ -\sqrt{3}(B_nA_m \pm A_nB_m) & n+m=odd \end{Bmatrix} \qquad (284)$$

where $r+1=4\equiv 1$ and $r-1=0\equiv 3$. For $s=r$, equation (281) gives

$$\widetilde{N}^{rr}_{nm} = \widetilde{N}^{r,r}_{nm}+\widetilde{N}^{r,-r}_{nm} = -\frac{(-)^n}{3(n+m)}\begin{Bmatrix} B_nA_m + A_nB_m \\ 0 \end{Bmatrix} - \frac{(-)^n}{3(n-m)}\begin{Bmatrix} B_nA_m - A_nB_m \\ 0 \end{Bmatrix} \qquad (285)$$

where the upper row refers to $n+m=even$ and the lower row refers to $n+m=odd$. Thus for $n+m=odd$, equation (285) gives

$$\widetilde{N}^{rr}_{2n+12m} = \widetilde{N}^{rr}_{2m2n+1} = 0, \qquad n,m\geq 0, r=1,2,3 \qquad (286)$$

For $n+m=even$, we have two cases to consider (1) $n=even$, $m=even$ and (2) $n=odd$, $m=odd$. For $n=even$, $m=even$ and $n\neq m$, equation (285) yields

$$\widetilde{N}^{rr}_{2n\ 2m} = -\frac{(-1)^{n+m}}{3}\left[\frac{b_{2n}a_{2m}+a_{2n}b_{2m}}{2n+2m} + \frac{b_{2n}a_{2m}-a_{2n}b_{2m}}{2n-2m}\right] \qquad (287)$$

For $n=odd$, $m=odd$ and $n\neq m$, equation (285) yields

$$\widetilde{N}^{rr}_{2n+1\ 2m+1} = \frac{(-1)^{n+m}}{3}\left[\frac{b_{2n+1}a_{2m+1}+a_{2n+1}b_{2m+1}}{(2n+1+2m+1)} + \frac{b_{2n+1}a_{2m+1}-a_{2n+1}b_{2m+1}}{(2n+1-2m+1)}\right] \qquad (288)$$

For $n=0$ or $m=0$, but $n+m=even\neq 0$, equation (285) yields

$$\widetilde{N}^{rr}_{02m} = -\frac{2}{3.2m}B_{2m} = -\frac{2(-)^m}{3.2m}b_{2m}, \qquad m>0, \quad r=1,2,3 \qquad (289)$$

$$\widetilde{N}^{rr}_{2n0} = -\frac{2}{3.2n}B_{2n} = -\frac{2(-)^n}{3.2n}b_{2n}, \qquad n>0, \quad r=1,2,3 \qquad (290)$$

Note that in this case we have $\widetilde{N}^{rr}_{02n} = \widetilde{N}^{rr}_{2n0}$ as expected.

Next we consider the case where $r\neq s$. First consider $\widetilde{N}^{12}_{nm}$; that is,

$$\widetilde{N}^{12}_{nm} = \widetilde{N}^{1,2}_{nm} + \widetilde{N}^{1,-2}_{nm} \qquad (291)$$

Substituting equation (283)in the right hand side of the above equation, we obtain

$$\widetilde{N}^{12}_{nm} = \widetilde{N}^{1,2}_{nm}+\widetilde{N}^{1,-2}_{nm} = \frac{1}{6(n+m)}\begin{Bmatrix} (-)^n(B_nA_m \pm A_nB_m) \\ \sqrt{3}(B_nA_m - A_nB_m) \end{Bmatrix} + \frac{1}{6(n-m)}\begin{Bmatrix} (-)^n(B_nA_m - A_nB_m) \\ \sqrt{3}(B_nA_m + A_nB_m) \end{Bmatrix} \qquad (292)$$

where the upper row refers to $n+m = even$ and the lower row refers to $n+m = odd$. For the non diagonal elements the above equation yields

$$\widetilde{N}^{12}_{2n2m} = \frac{(-1)^{n+m}}{6}\left[\frac{b_{2n}a_{2m} - a_{2n}b_{2m}}{2n+2m} + \frac{b_{2n}a_{2m} - a_{2n}b_{2m}}{2n-2m}\right], \qquad n, m > 0, \tag{293}$$

$$\widetilde{N}^{12}_{2n2m+1} = \frac{(-1)^{n+m}}{2\sqrt{3}}\left[\frac{b_{2n}a_{2m+1} - a_{2n}b_{2m+1}}{2n+(2m+1)} + \frac{b_{2n}a_{2m+1} + a_{2n}b_{2m+1}}{2n-(2m+1)}\right], \qquad n > 0\ ; m \geq 0, \tag{294}$$

$$\widetilde{N}^{12}_{2n+12m} = \frac{(-1)^{n+m}}{2\sqrt{3}}\left[\frac{b_{2n+1}a_{2m} - a_{2n+1}b_{2m}}{(2n+1)+2m} + \frac{b_{2n+2}a_{2m} + a_{2n+1}b_{2m}}{(2n+1)-2m}\right], \qquad n \geq 0;\ m > 0 \tag{295}$$

and

$$\widetilde{N}^{12}_{2n+12m+1} = \frac{(-1)^{n+m}}{6}\left[\frac{b_{2n+1}a_{2m+1} - a_{2n+1}b_{2m+1}}{(2n+1)+(2m+1)} + \frac{b_{2n+1}a_{2m+1} - a_{2n+1}b_{2m+1}}{(2n+1)-(2m+1)}\right], \qquad n, m \geq 0 \tag{296}$$

For $n = 0$ or $m = 0$, but $n + m = even \neq 0$, equation (292)

$$\widetilde{N}^{12}_{2n\ 0} = \frac{(1)^n}{3}\frac{b_{2n}}{2n}, \qquad \widetilde{N}^{12}_{0\ 2m} = \frac{(1)^m}{3}\frac{b_{2m}}{2m} \tag{297}$$

and

$$\widetilde{N}^{12}_{2n+1\ 0} = \frac{(1)^m}{\sqrt{3}}\frac{b_{2n+1}}{2n+1}, \qquad \widetilde{N}^{12}_{0\ 2m+1} = -\frac{(1)^m}{\sqrt{3}}\frac{b_{2m+1}}{2m+1} \tag{298}$$

From the cyclic symmetry, it follows that

$$\widetilde{N}^{23}_{nm} = \widetilde{N}^{31}_{nm} = \widetilde{N}^{12}_{nm} \tag{299}$$

Next we consider the $\widetilde{N}^{21}_{nm}$ coefficient. In this case combining equations (284) and (281) we obtain

$$\widetilde{N}^{21}_{nm} = \widetilde{N}^{2,1}_{nm} + \widetilde{N}^{2,-1}_{nm} = \frac{1}{6}\left[\frac{1}{(n-m)}\begin{Bmatrix}(-)^n (B_n A_m - A_n B_m) \\ -\sqrt{3}\,(B_n A_m + A_n B_m)\end{Bmatrix} + \frac{1}{(n+m)}\begin{Bmatrix}(-)^n (B_n A_m - A_n B_m) \\ -\sqrt{3}\,(B_n A_m - A_n B_m)\end{Bmatrix}\right] \tag{300}$$

where the upper row refers to $n+m = even$ and the lower row refers to $n+m = odd$. For the non diagonal elements the above equations yields

$$\begin{aligned}\widetilde{N}^{21}_{2n\ 2m} &= \frac{(-1)^{n+m}}{6}\left[\frac{b_{2n}a_{2m} - a_{2n}b_{2m}}{2n-2m} + \frac{b_{2n}a_{2m} + a_{2n}b_{2m}}{2n+2m}\right] \\ &= \widetilde{N}^{12}_{2n\ 2m}, \qquad n, m > 0,\end{aligned}$$

$$\begin{aligned}\widetilde{N}^{21}_{2n\ 2m+1} &= -\frac{(-)^{n+m}}{2\sqrt{3}}\left[\frac{b_{2n}a_{2m+1} + a_{2n}b_{2m+1}}{2n-(2m+1)} + \frac{b_{2n}a_{2m+1} - a_{2n}b_{2m+1}}{2n+(2m+1)}\right] \\ &= -\widetilde{N}^{12}_{2n2m+1}, \qquad n > 0, m \geq 0,\end{aligned} \tag{301}$$

$$
\begin{aligned}
\widetilde{N}^{21}_{2n+1\ 2m} &= -\frac{(-)^{n+m}}{2\sqrt{3}}\left[\frac{b_{2n+1}a_{2m}+a_{2n+1}b_{2m}}{(2n+1)-2m}+\frac{b_{2n+2}a_{2m}-a_{2n+1}b_{2m}}{(2n+1)+2m}\right] \\
&= -\widetilde{N}^{12}_{2n+1\ 2m}, \quad n\geq 0, m>0 \qquad (302)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{N}^{21}_{2n+1\ 2m+1} &= -\frac{(-)^{n+m}}{6}\left[\frac{b_{2n+1}a_{2m+1}-a_{2n+1}b_{2m+1}}{(2n+1)-(2m+1)}+\frac{b_{2n+1}a_{2m+1}+a_{2n+1}b_{2m+1}}{(2n+1)+(2m+1)}\right] \\
&= \widetilde{N}^{12}_{2n+1\ 2m+1}, \quad n,m\geq 0 \qquad (303)
\end{aligned}
$$

From the cyclic symmetry it follows that $\widetilde{N}^{32}_{nm}=\widetilde{N}^{13}_{nm}=\widetilde{N}^{21}_{nm}$. Likewise the zero nonzero elements can be computed by combining equations (284) and (281). Skipping a rather simple algebra we find

$$
\widetilde{N}^{21}_{0\ 2m}=\frac{(-)^m}{3}\frac{b_m}{2m}=\widetilde{N}^{12}_{0\ 2m}, \qquad \widetilde{N}^{21}_{2n\ 0}=\frac{(-)^n}{3}\frac{b_{2n}}{2n}=\widetilde{N}^{12}_{2n\ 0} \qquad (304)
$$

where $n,m>0$ and

$$
\widetilde{N}^{21}_{0\ 2m+1}=\frac{(-)^m}{\sqrt{3}}\frac{b_{2m+1}}{(2m+1)}, \qquad \widetilde{N}^{21}_{2n+1\ 0}=-\frac{(-)^n}{\sqrt{3}}\frac{b_{2n+1}}{2n+1}=-\widetilde{N}^{12}_{2n+1\ 0} \qquad (305)
$$

where $n,m\geqslant 0$.Once more we see that

$$
\widetilde{N}^{32}_{nm}=\widetilde{N}^{13}_{nm}=\widetilde{N}^{21}_{nm} \qquad (306)
$$

At this point, it is worth emphasizing that for $n+m=\mathit{even}$ we have $\widetilde{N}^{23}_{nm}=\widetilde{N}^{31}_{nm}=\widetilde{N}^{12}_{nm}=\widetilde{N}^{32}_{nm}=\widetilde{N}^{13}_{nm}=\widetilde{N}^{21}_{nm}$. Thus for the diagonal elements it suffice to compute $\widetilde{G}^{12}_{2n\ 2n}$ and $\widetilde{G}^{12}_{2n+1\ 2n+1}$ and all others follow from the cyclic symmetry in the string index. These two can be computed by taking the limit of $m\longrightarrow n$ in equation (293) and (296) respectively.

In the limit of $m\longrightarrow n$, equation (293) gives

$$
\begin{aligned}
\widetilde{N}^{12}_{2n\ 2n} &= \frac{1}{6}\frac{2a_{2n}b_{2n}}{2\cdot 2n}+\lim_{m\longrightarrow n}\frac{(-)^{n+m}}{6}\frac{b_{2n}a_{2m}-a_{2n}b_{2m}}{2n-2m} \qquad (307) \\
&= \frac{1}{6}\frac{a_{2n}b_{2n}}{2n}-\frac{1}{6}\frac{\sqrt{3}}{\pi}\left(a_{2n}\widetilde{O}^{b}_{2n}-b_{2n}\widetilde{O}^{a}_{2n}\right) \qquad (308)
\end{aligned}
$$

where we have used equation (217) to evaluate the limit. Now from the cyclic symmetry it follows that

$$
\begin{aligned}
\widetilde{N}^{12}_{2n\ 2n} &= \widetilde{N}^{23}_{2n\ 2n}=\widetilde{N}^{31}_{2n\ 2n}=\widetilde{N}^{13}_{2n\ 2n}=\widetilde{N}^{32}_{2n\ 2n}=\widetilde{N}^{21}_{2n\ 2n} \\
&= \frac{1}{6}\frac{a_{2n}b_{2n}}{2n}-\frac{1}{6}\frac{\sqrt{3}}{\pi}\left(a_{2n}\widetilde{O}^{b}_{2n}-b_{2n}\widetilde{O}^{a}_{2n}\right) \qquad (309)
\end{aligned}
$$

For the odd diagonal elements, in the limit of $m\longrightarrow n$, equation (296) gives

$$
\begin{aligned}
\widetilde{N}^{12}_{2n+1\ 2m+1} &= -\frac{1}{6}\frac{2a_{2n+1}b_{2n+1}}{2.\,(2n+1)}-\frac{1}{6}\lim_{m\longrightarrow n}(-)^{n+m}\frac{b_{2n+1}a_{2m+1}-a_{2n+1}b_{2m+1}}{(2n+1)-(2m+1)} \\
&= \frac{1}{6}\frac{a_{2n+1}b_{2n+1}}{2n+1}+\frac{1}{6}\frac{\sqrt{3}}{\pi}\left(a_{2n+1}\widetilde{E}^{b}_{2n+1}-b_{2n+1}\widetilde{E}^{a}_{2n+1}\right) \qquad (310)
\end{aligned}
$$

where we have used equation (209) to evaluate the limit. Once more from the cyclic symmetry it follows that

$$\begin{aligned}
\widetilde{N}^{12}_{2n+1\ 2n+1} &= \widetilde{N}^{23}_{2n+1\ 2n+1} = \widetilde{N}^{31}_{2n+1\ 2n+1} = \widetilde{N}^{13}_{2n+1\ 2n+1} = \widetilde{N}^{32}_{2n+1\ 2n+1} = \widetilde{N}^{21}_{2n+1\ 2n+1} \\
&= \frac{1}{6}\frac{a_{2n+1}b_{2n+1}}{2n+1} + \frac{1}{6}\frac{\sqrt{3}}{\pi}\left(a_{2n+1}\widetilde{E}^b_{2n+1} - b_{2n+1}\widetilde{E}^a_{2n+1}\right) \qquad (311)
\end{aligned}$$

To complete the computation of the Neumann coefficients, we still need to calculate the diagonal elements for $r = s$. In this case (285) and the identities in (209) and (217) yield

$$\begin{aligned}
\widetilde{N}^{rr}_{2n+1\ 2n+1} &= \lim_{m\longrightarrow n} \widetilde{N}^{rr}_{2n+1\ 2m+1} \\
&= \frac{1}{3}\lim_{m\longrightarrow n}(-)^{n+m}\left[\frac{b_{2n+1}a_{2m+1} + a_{2n+1}b_{2m+1}}{(2n+1)+(2m+1)} + \frac{b_{2n+1}a_{2m+1} - a_{2n+1}b_{2m+1}}{(2n+1)-(2m+1)}\right] \qquad (312) \\
&= \frac{1}{3}\frac{a_{2n+1}b_{2n+1}}{2n+1} - \frac{1}{\pi\sqrt{3}}\left(a_{2n+1}\widetilde{E}^b_{2n+1} - b_{2n+1}\widetilde{E}^a_{2n+1}\right) \qquad (313)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{N}^{rr}_{2n\ 2n} &= \lim_{m\longrightarrow n}\widetilde{N}^{rr}_{2n\ 2m} = \lim_{m\longrightarrow n}(-)^{n+m}\left[-\frac{(b_{2n}a_{2m} + a_{2n}b_{2m})}{3\,(2n+2m)} - \frac{(b_{2n}a_{2m} - a_{2n}b_{2m})}{3\,(2n-2m)}\right] \\
&= -\frac{1}{3}\frac{a_{2n}b_{2n}}{2n} + \frac{1}{\pi\sqrt{3}}\left(a_{2n}\widetilde{O}^b_{2n} - b_{2n}\widetilde{O}^a_{2n}\right) \qquad (314)
\end{aligned}$$

This completes the computation of the $\widetilde{N}^{rs}_{nm}$ coefficients.

C The $C^{[r,n]}_{qk}$ Coefficients

In this appendix we give the first few values of the coefficients $C^{[r,n]}_{qk}$ as calculated from the the recursion relations in (141) and (142). In Table C.1 we give the first few values of the coefficients $C^{[-1]}_{qk}$ in terms of the $\overline{\Lambda}^r_n$ as found from equations (141) and (142). We also give in table C.2 the first few numerical values of the coefficients $C^{[-1]}_{qk}$ as calculated from equation (141) and (142).

$C^{[r,-1]}_{qk}$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$q=0$	1					
$q=1$	$-\frac{6}{11}\sqrt{3}-\frac{8}{11}$	$\frac{6}{11}\sqrt{3}+\frac{8}{11}$				
$q=2$	$\frac{5}{11}$	0	$\frac{5}{11}$			
$q=3$	$\frac{12\,960}{171\,457}\sqrt{3}+\frac{10\,240}{171\,457}$	0	$-\frac{30}{121}\sqrt{3}-\frac{40}{121}$	$\frac{29\,550}{171\,457}\sqrt{3}+\frac{46\,440}{171\,457}$		
$q=4$	$\frac{3317\,760}{81\,099\,161}\sqrt{3}-\frac{4112\,144}{81\,099\,161}$	0	$-\frac{480}{1331}\sqrt{3}-\frac{585}{1331}$	$\frac{1030\,080}{1886\,027}\sqrt{3}+\frac{1806\,840}{1886\,027}$	$-\frac{12\,960}{57\,233}\sqrt{3}-\frac{26\,773}{57\,233}$	
$q=5$	$-\frac{8907\,155\,740\,608}{1078\,645\,685\,122\,291}\sqrt{3}-\frac{30\,264\,361\,046\,016}{1078\,645\,685\,122\,291}$	0	$-\frac{7364\,100}{20\,746\,297}\sqrt{3}-\frac{15\,193\,840}{20\,746\,297}$	$\frac{25\,372\,020}{20\,746\,297}\sqrt{3}+\frac{44\,385\,840}{20\,746\,297}$	$-\frac{1090\,198\,368}{892\,090\,771}\sqrt{3}-\frac{1711\,928\,864}{892\,090\,771}$	$\frac{1351\,278\,627\,9}{5393\,228\,425\,6} + \frac{5945\,189\,91}{16\,179\,685\,2}$

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