2012 Hawaii University International Conferences
Education, Math & Engineering Technology
July 31st to August 2nd
Waikiki Beach Marriot Resort & Spa
Honolulu, Hawaii

The Mathematical Magic the Mulatu Numbers

Mulatu Lemma, Jonathan Lambright, and Spyros Andreou
Department of Engineering
Technology and Mathematics
Savannah State University
Savannah, GA 31404, U.S.A
2012 Hawaii University International Conference

The Mathematical Magic the Mulatu Numbers

Mulatu Lemma, Jonathan Lambright, and Spyros Andreou

Department of Engineering
Technology and Mathematics

Savannah State University
Savannah, GA 31404, U.S.A
2012 Hawaii University International Conference

Abstract: The Mulatu numbers were studied in [1] and [2]. The numbers are sequences of numbers of the form: **4,1,5,6,11,17,28,45...** The numbers have wonderful and amazing properties and patterns.

In mathematical terms, the sequence of the Mulatu numbers is defined by the following recurrence relation:

$$M_n := \begin{cases} 4 & \text{if } n=0; \\ 1 & \text{if } n=1; \\ M_{n-1}+M_{n-2} & \text{if } n>1. \end{cases}$$

In [1] and [2] some properties and patterns of the numbers were considered. In this paper, we investigate additional properties and patterns of these fascinating numbers. Many beautiful mathematical identities involving the Mulatu numbers in relation with the Fibonacci numbers and the Lucas numbers will be more explored.

2000 Mathematical Subject Classification: 11

Key Words: Mulatu numbers, Mulatu sequences, Fibonacci numbers, Lucas numbers, Fibonacci sequences, and Lucas sequences.

1. **Introduction and Background**. As given in [1] and[2], the Mulatu sequence has wealthy mathematical properties and patterns like the two celebrity sequences of Fibonacci and Lucas.

In this paper, more interesting relationships of the Mulatu numbers to the Fibonacci and Lucas numbers will be presented.

Here are the First 21 Mulatu, Fibonacci, and Lucas numbers for quick reference.

Mulatu(M_n), **Fibonacci**(F_n) and **Lucas**(L_n) **Numbers**
(Tables 1 & 2)

Table 1

n:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
$M_{n:}$	4	1	5	6	11	17	28	45	73	118	191	309
F_n:	0	1	1	2	3	5	8	13	21	34	55	89
L_n:	2	1	3	4	7	11	18	29	47	76	123	199

Table 2

n:	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
M_n	500	809	1309	2118	3427	5545	8972	14517	23489
F_n:	144	233	377	610	987	1597	2584	4181	6765
L_n:	322	521	843	1364	2207	3571	5778	9349	15127

Remark 1 : Throughout this paper M, F, and L stand for Mulatu numbers, Fibonacci numbers, and Lucas number respectively.

The following well-known identities of Mulatu numbers [1], Fibonacci and Lucas numbers are required in this paper and hereby listed for quick reference.

(1) $L_n = F_{n-1} + F_{n+1}$

(2) $F_{n+1} = F_n + F_{n-1}$

(3) $M_n = L_n + 2F_{n-1}$.

(4) $F_{2n} = F_n L_n$

(5) $5F^2{}_n - L_n{}^2 = 4(-1)^{n+1}$

(6) $F_n = \dfrac{L_{n+1} + L_{n-1}}{5}$

(7) $L_{n+1} = L_n + L_{n-1}$

(8) $F_{n+k} = F_{n-1}F_k + F_n F_{k+1}$

The Main Results.

Theorem 1. Some Divisibility Properties of M.

(a) If M_n is divisible by 2, then $M^2_{n+1} - M^2_{n-1}$ is divisible by 4

(b) If M_n is divisible by 3, then $M^3_{n+1} - M^3_{n-1}$ is divisible by 9.

Proof: We repeatedly use the relation $M_{n+1} = M_n + M_{n-1}$ in our proof:

(a) $M^2_{n+1} - M^2_{n-1}$

$$= (M_{n+1} - M_{n-1})(M_{n+1} + M_{n-1}) = M_n\,(M_n + M_{n-1} + M_{n-1}) = M^2_n + 2M_nM_{n-1}.$$

Now it is easy to see that if M_n is divisible by 2, then $M^2_{n+1} - M^2_{n-1}$ is divisible by 4

(b) $M^3_{n+1} - M^3_{n-1} = (M_{n+1} - M_{n-1})(M^2_{n+1} + M_nM_{n-1} + M^2_{n-1})$

$$= M_n\,(M^2_{n+1} + M_{n+1}M_{n-1} + M^2_{n-1})$$

$$= M_n((M_n + M_{n-1})^2 + M_{n-1}(M_n + M_{n-1}) + M^2_{n-1})$$

$$= M_n(M^2_n + 3M_nM_{n-1} + 3M^2_{n-1})$$

$$= M^3_n + 3M^2_nM_{n-1} + 3M_nM^3_{n-1}$$

Hence M_n is divisible by 3 $\Rightarrow$ M^3_{n+1} - M^3_{n-1} is divisible by 9.

Theorem 2. (Expressing M in terms of F)

Let M_n and F_n be any Mulatu and Fibonacci Numbers respectively. Then we have:

$$M_n = F_{n-3} + F_{n-1} + F_{n+2}$$

Proof: We use induction on n.

(1) When n = 0, the formula is true as $M_0 = F_{-3} + F_{-1} + F_2$ and using $F_{-n} = (-1)^{n+1}F_n$, we have 4= 2+1+1= 4.

(2) Assume the formula is true for n =1,2,3…k-1, k.

(3) Verify the formula for n= k+1.

Note that

$$M_{k+1} = M_k + M_{k-1} = F_{k-3} + F_{k-1} + F_{k+2} + F_{k-4} + F_{k-2} + F_{k+1}$$

$$\Rightarrow \quad M_k + M_{k-1} = F_{k-4} + F_{k-3} + F_{k-2} + F_{k-1} + F_{k+2} + F_{k+1}$$

$$= \; F_{k-2} + F_k + F_{k+3}$$

Hence by Induction, the theorem follows.

Example 1: Take n=6. Note that $M_6 = 28$, $F_3 = 2, F_5 = 5, and F_8 = 21.$

Hence $M_6 = F_3 + F_5 + F_8$

Corollary 1.

$$M_n = L_n + 2F_{n-1}.$$

Proof: By **Theorem 2**, we have $M_n = F_{n-3} + F_{n-1} + F_{n+2}$. Using the following Fibonacci- Lucas sequences' relations

(1) $F_{n-3} = F_{n-1} - F_{n-2}$

(2) $F_{n+2} = F_{n+1} + F_n$

(3) $L_n = F_{n-1} + F_{n+1}$

we have,

$$F_{n-3} + F_{n-1} + F_{n+2} = F_{n-1} - F_{n-2} + F_{n-1} + F_{n+1} + F_n =$$
$$F_{n-1} - F_{n-2} + F_{n-1} + F_{n+1} + F_{n-1} + F_{n-2} = F_{n-1} + F_{n-1} + L_n.$$

$$\Rightarrow F_{n-3} + F_{n-1} + F_{n+2} = L_n + 2F_{n-1}.$$

$$\Rightarrow M_n = L_n + 2F_{n-1}.$$

Example 2: Take n= 6. Note that $M_6 = 28$ **,** $L_6 = 18$ & $F_5 = 5$ **.**

Thus, $M_6 = L_6 + \mathbf{2} F_5$

Theorem 3. (Expressing M in terms of L)

Let M_n **and** L_n be any Mulatu and Lucas Numbers respectively. Then we have:

$$M_n = \frac{7L_n + 2L_{n-2}}{5}$$

Proof: The theorem easily follows from Corollary 1, using the Fibonacci-Lucas relation

$$F_n = \frac{L_{n+1} + L_{n-1}}{5}$$

Example 3: Let n=7. Note that:

$M_7 = 45, L_7 = 29$ & $L_5 = 11.$

We have, $\frac{7(29) + 2(11)}{5} = 45 = M_7.$

The Fibonacci and Lucas Numbers have addition formulas. A question may be asked if M has also an addition formula. The answer is positive and produces the following theorem.

Theorem 4. **<u>The addition formula *for* Mulatu numbers</u>**.

$$M_{n+k} = F_{n-1} M_k + F_n M_{k+1}$$

Proof: By Theorem 2, we have,

$$M_n = F_{n-3} + F_{n-1} + F_{n+2} .$$

Hence it follows that

$$M_{n+k} = F_{n+k-3} + F_{n+k-1} + M_{n+k+2}.$$

Now using the addition formula for Fibonacci numbers given above, it follows that

$$M_{n+k} = (F_{n-1}F_{k-3} + F_n\ F_{k-2}) + (F_{n-1}F_{k-1} + F_nF_k) + (F_{n-1}F_{k+2} + F_nF_{k+3})$$

$$= (F_{n-1}F_{k-3} + F_{n-1} + \mathrm{F}_{\mathrm{k}-1} + F_{n-1}F_{k+2}) + (\mathrm{F}_n\,\mathrm{F}_{k-2} + \mathrm{F}_n\ F_k + F_nF_{k+3})$$

$$= F_{n-1}(F_{k-3} + \mathrm{F}_{\mathrm{k}-1} + F_{k+2}) + F_n(\mathrm{F}_{k-2} + F_k +_n F_{k+3})$$

$$= F_{n-1}\ M_k + F_n\ M_{k+1}.$$

Hence the theorem is proved.

Example4: Let n=4 and k=3. Then we have: $M_7 = M_{4+3}$**.** Note that

$F_3 = 2, F_4 = 3, M_3 = 6, \& M_4 = 11.$ Hence, $F_3M_4 + F_4M_4 = 2(6) + 3(11) = 45 = M_7$

Corollary 2:

$$M_{2n-1} = F_{2n} - 3F^2{}_{n-1} + 6F_nF_{n-1}$$

Proof: By Theorem 4 we have,

$$M_{2n-1} = M_{n+(n-1)} = F_{n-1}M_{n-1} + F_nM_n$$

$$= F_{n-1}M_{n-1} + F_n(L_n + 2F_{n-1})$$

$$= F_{n-1}M_{n-1} + F_nL_n + 2F_nF_{n-1}$$

$$= F_{n-1}M_{n-1} + F_{2n} + 2F_nF_{n-1}.$$

Now applying *Theorem 4* to M_{n-1}, we have

$$M_{n-1} = M_{(n-1)+0} = F_{n-2}M_0 + F_{n-1}M_1 = 4F_{n-2} + F_{n-1} \text{ and}$$

$$4F_{n-2} + F_{n-1} = 4(F_n - F_{n-1}) + F_{n-1} = -3F_{n-1} + 4F_n.$$

Hence, $M_{2n-1} = F_{2n} + F_{n-1}(-3F_{n-1} + 4F_n) + 2F_nF_{n-1} = F_{2n} - 3F^2{}_{n-1} + 6F_nF_{n-1}$

Example 5. Let n = 5. Then $M_{2n-1} = M_9$. Note that:

(1) $F_{2n} = F_{10} =$ 55 (2) -3 $F^2{}_4$= -3(9) = -27 (3) $6F_5F_4$=6(5)(3)= 90.

We have, 55-27+90=118=M_9.

Corollary 3. The Subtraction formula for **Mulatu** numbers

$$M_{n-k} = 4F_{n-k+1} - 3F_{n-k}$$

Proof: $M_{n-k} = M_{(n-k)+0}$ and hence by Theorem 4, we have

$$M_{n-k} = F_{n-k-1}M_0 + F_{n-k}M_1$$

$$= 4F_{n-k-1} + F_{n-k}$$

$$= 4(F_{n-k-1} + F_{n-k}) - 3F_{n-k}$$

$$= 4F_{n-k+1} - 3F_{n-k}.$$

Example 6.

Let n= 8 and k=3. Then we have: $M_{8-3} = M_5$. Note that $4F_6 = 4(8)\ \&\ 3(F_5) = 15$ Hence, $4F_6 - 3F_5 =$ 32-15=17= M_5.

Lemma1.

F_{2n} - $M_n\,\mathrm{F}_{n+1} - F_{n+1}F_n = -L^2{}_n$

Proof: We use the identities listed above to prove the theorem.

Note that F_{2n} - $M_n\,\mathrm{F}_{n+1} - F_{n+1}F_n = F_nL_n - M_n\,\mathrm{F}_{n+1} - F_{n+1}F_n$

$$= F_n(F_{n-1} + F_{n+1}) - F_{n+1}(L_n + 2F_{n-1}) - F_{n+1}F_n$$

$$= F_n(F_{n-1} + F_{n+1}) - (F_n + F_{n-1})(L_n + 2F_{n-1}) - F_{n+1}F_n$$

$$= F_n(F_{n-1} + F_{n+1}) - (F_n + F_{n-1})(\mathrm{F}_{n+1} + F_{n-1} + 2F_{n-1}) - (F_n + F_{n-1})F_n$$

$$=F_n(F_{n-1}+F_n+F_{n-1})-$$

$$(F_n+F_{n-1})(\mathrm{F_n}+F_{n-1}+F_{n-1}+2F_{n-1})-(F_n+F_{n-1})F_n$$

$$=F_n(2F_{n-1}+F_n)-(F_n+F_{n-1})(\mathrm{F_n}+4F_{n-1})-(F_n+F_{n-1})F_n$$

$$=2F_nF_{n-1}+F^2{}_n-F^2{}_n-4F_nF_{n-1}-F_nF_{n-1}-4F_{n-1}{}^2-F^2{}_n-F_{n-1}F_n$$

$$=-F^2{}_n-4F_nF_{n-1}-4F^2{}_{n-1}$$

$$=-(F^2{}_n+4F_nF_{n-1}+4F^2{}_{n-1})$$

$$=-\left(F_n+2F_{n-1}\right)^2$$

$$=-\left(F_n+F_{n-1}+F_{n-1}\right)^2$$

$$=-(F_{n+1}+F_{n-1})^2$$

$$=-L^2{}_n$$

Example 7. Let n =5. Then if follows that:

$$F_{2n}-M_nF_{n+1}-F_{n+1}F_n=F_{10}-M_5F_6-F_6F_5=55-136-40=-121=-L_5^2$$

The following result deals with the Double -angle type formula. It is rather an amazingly interesting strong result.

The Lucas Numbers and Fibonacci Numbers have the following double angle like formulas.

1. $L_{2n}=\frac{1}{2}\left(5F_n^2+L_n^2\right),$
2. $F_{2n}=F_nL_n$

What about the Mulatu Numbers? The Answer is given by the following Theorem.

Theorem 6. Fundamental identity (Double Angle Like Theorem For Mulatu Numbers).

$M_{2n}=M_nL_n+4(-1)^{n+1}$

Proof: By *Theorem 4*, $M_{2n}=M_{n+n}=\mathrm{F}_{n-1}M_n+F_nM_{n+1}$. Again applying ***Theorem 4***, to M_{n+1} and using $L_n=F_{n+1}+F_{n-1}$, we get

$$M_{2n}=\mathrm{F}_{n-1}M_n+F_n(F_{n-1}M_1+F_nM_2)$$

$$=\mathrm{F}_{n-1}M_n+F_n(F_{n-1}+5F_n).$$

$$=\mathrm{F}_{n-1}M_n+F_nF_{n-1}+5F^2{}_n.$$

$= (L_n - F_{n+1})M_n + F_nF_{n-1} + 5F^2{}_n$

$= L_nM_n - F_{n+1} M_n + F_nF_{n-1} + 5F^2{}_n$

$= L_nM_n - (F_n + F_{n-1})(L_n + 2F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$

$= L_nM_n - (F_n + F_{n-1})(F_{n+1} + F_{n-1} + 2F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$

$= L_nM_n - (F_n + F_{n-1})(F_n + 4F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$

$= L_nM_n - F^2{}_n - 4F_nF_{n-1} - F_nF_{n-1} - 4F^2{}_{n-1} + F_nF_{n-1} + 5F^2{}_n$

$= L_nM_n - F^2{}_n - 4F_nF_{n-1} - 4F^2{}_{n-1} + 5F^2{}_n$

$= L_nM_n - (F^2{}_n + 4F_nF_{n-1} + 4F^2{}_{n-1}) + 5F^2{}_n$

From the proof of Lemma 1, we know that $F^2{}_n + 4F_nF_{n-1} + 4F^2{}_{n-1} = L^2{}_n$. Hence $M_{2n} = L_nM_n - L^2{}_n + 5F^2{}_n$. Now using that $5F^2{}_n - L^2{}_n = 4(-1)^{n+1}$ from above, it easily follows that $M_{2n} = L_nM_n + 4(-1)^{n+1}$.

Example 8:

Let n=5**.** Then if follows that $M_{2n} = M_{10} = 191.$ Also we have,

$L_nM_n + 4(-1)^{n+1} = L_5M_5 + 4(-1)^6 = 11(17) + 4 = 191 = M_{10}$

Corollary 4: $M_{2n} = L^2{}_n + 4F^2{}_{n-1} + 2F_nF_{n-1} + 4(-1)^{n+1}$

Proof: We have $M_{2n} = M_nL_n + 4(-1)^{n+1}$

$= (L_n + 2F_{n-1}) L_n + 4(-1)^{n+1}$

$= L^2{}_n + 2F_{n-1}L_n + 4(-1)^{n+1}$

$= L^2{}_n + 2F_{n-1}(F_{n+1} + F_{n-1}) + 4(-1)^{n+1}$

$= L^2{}_n + 2F_{n-1}(F_n + F_{n-1} + F_{n-1}) + 4(-1)^{n+1}$

$= L^2{}_n + 4F^2{}_{n-1} + 2F_nF_{n-1} + 4(-1)^{n+1}$

Example 9: Let n =6. Then we have. $M_{12} = 500$. Note that:

$4F^2{}_{n-1} + 2F_nF_{n-1} + 4(-1)^{n+1} + L^2{}_n =$

$4F^2{}_5 + 2F_6F_5 + 4(-1)^7 + L^2{}_6 =$

4(25)+2(8)5-4+324=500= M_{12}

Some Interesting Open Questions.

(1) Are there any more triangular numbers in Mulatu numbers other than 1, 6, 28, and 45? If so, are they finite or infinite?

(2) Are there any more Fermat numbers in Mulatu numbers other than 5 and 17? If so, are they finite or infinite?

(3) Are there any more Fibonacci numbers in Mulatu numbers other than 1 and 5? If so, are they finite or infinite?

(4) Are there any more Lucas numbers in Mulatu numbers other than 1 and 11? If so, are they finite or infinite?

(5) Observe that for n= 1,6,11, 16, and 21 all M, F, and L numbers have the same last digit. Is this pattern finite or infinite?

Acknowledgement:

We would like to thank the PRISM Grant of NSF and the Department of Engineering Technology and Mathematics at SSU for the great support provided they provided for us.

References:

1. Mulatu Lemma, The Mulatu Numbers, *Advances and Applications in Mathematical Sciences,* Volume 10, issue 4,august 2011, page 431-440.

2. Mulatu Lemma, The Fascinating Mathematical Beauty of the Mulatu Numbers, *Advances and Applications in Mathematical Sciences,* Volume 10, issue 4, November t 2011, page 431-440..