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# THE COMPUTATION OF THE MOBIUS FUNCTION OF A MOBIUS

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*THE COMPUTATION OF THE MÖBIUS FUNCTION OF A MÖBIUS  
CATEGORY*

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ABSTRACT. The paper presents some results for reducing the computation of the Möbius function of a Möbius category that arises from a combinatorial inverse semigroup to that of locally finite partially ordered sets. We illustrate the computation of the Möbius function with an example.

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1. INTRODUCTION (MÖBIUS CATEGORIES)

The theory of Möbius functions in categories was initiated and developed by Leroux and collaborators such as Content and Lemay (see [1],[6],[7]), and in the same time by Haigh [2]. Recently, Leinster [5], Lawvere and Menni [3] have brought attention to the problem of Möbius inversion in categories in a broader context. In previous papers [9]-[13], the first author of the present paper has found connections between the theory of combinatorial inverse semigroups and the theory of Möbius categories. The combinatorial inverse semigroups provide special examples of Möbius categories.

A Möbius category  $C$  (in the sense of Leroux) is a small category satisfying the following conditions:

(1) any morphism  $f \in \text{Mor}C$  has only a finitely many non-trivial factorizations  $f = gh$ ;

(2) an incidence function  $\xi : \text{Mor}C \rightarrow \mathbb{C}$  has a convolution inverse if and only if  $\xi(f) \neq 0$  for each identity morphism  $f$  of  $C$ . (The convolution  $*$  of two incidence functions  $\xi$  and  $\eta$  is defined by:  $(\xi * \eta)(f) = \sum_{f=gh} \xi(g)\eta(h)$ , and the convolution identity is  $\delta$  given by:  $\delta(f) = 1$  if  $f$  is an identity morphism and  $\delta(f) = 0$  otherwise.)

The Möbius function  $\mu$  of a Möbius category  $C$  is the convolution inverse of the zeta function  $\zeta$ :  $\zeta(f) = 1$  for any morphism  $f$  of  $C$ . The Möbius inversion formula is then nothing but the statement:  $\xi = \eta * \zeta \Leftrightarrow \eta = \xi * \mu$ . This is also the Möbius inversion formula in number theory, the functions being arithmetic functions and the convolution  $*$ , the Dirichlet product.

Now, a partially ordered set (poset)  $(P, \leq)$  as a category (the objects are the elements of  $P$ , and there is a morphism  $x \rightarrow y$  if and only if  $x \leq y$ ) is Möbius if and only if it is locally finite (i.e. any interval in  $P$  is finite). So, Möbius inversion, a useful tool in number theory, was generalized to categories via Rota's theory of Möbius functions. The computation of poset's Möbius function has a central place in Rota's theory ([8]). There are many remarkable fruitful methods for computing poset's Möbius functions.

Our main interest here is in the Möbius function for Möbius categories that arises from combinatorial inverse semigroups. There are some approaches to compute such Möbius functions using poset's Möbius functions. In Section 3 an example is considered to illustrate one of the presented methods.

## 2. LINKS WITH INVERSE SEMIGROUPS AND POSET'S MÖBIUS FUNCTIONS

A semigroup  $S$  is an inverse semigroup if every element  $s \in S$  has a unique inverse  $s^{-1}$ , in the sense that  $ss^{-1}s = s$  and  $s^{-1}ss^{-1} = s^{-1}$ . An inverse semigroup  $S$  possesses a natural partial order relation  $\leq$  defined by:  $s \leq t \Leftrightarrow s = ss^{-1}t$ . An inverse semigroup is locally finite if the poset of idempotents  $(E(S), \leq)$  is locally finite. Two elements  $s, t \in S$  are said to be  $\mathcal{D}$ -equivalent if and only if there exists an element  $x \in S$  such that  $s^{-1}s = x^{-1}x$  and  $xx^{-1} = tt^{-1}$ . A combinatorial semigroup is a semigroup whose all subgroups are trivial.

The natural partial order on inverse semigroups implies a combinatorial approach of inverse semigroups via Rota's Möbius inversion. The idea was explored successfully by Steinberg [14],[15].

The theory of Leech's ([4]) division categories open a new way to combinatorial approach of inverse semigroups, in this time, via categorical Möbius inversion. By [9, Theorem 3.3],

”the reduced division category  $C_F(S)$  relative to an idempotent transversal  $F$  of the  $\mathcal{D}$ -classes of an inverse monoid  $S$  with  $1 \in F$  is a Möbius category if and only if  $S$  is locally finite and combinatorial”.

This category  $C_F(S)$  is defined by:

- $ObC_F(S) = F$ ;
- $Hom_{C_F(S)}(e, f) = \{(s, e) | s \in S, s^{-1}s \leq e \text{ and } ss^{-1} = f\}$ ;
- The composition of two morphisms  $(s, e) : e \rightarrow f$  and  $(t, f) : f \rightarrow g$  is given by  $(t, f) \cdot (s, e) = (ts, e)$ .

The Möbius function  $\mu$  of this Möbius category  $C_F(S)$  is called the Möbius function of  $S$ . If  $S$  is with zero or has no identity element then a slight correction is needed to define the Möbius category of  $S$  (see [11]).

Starting with a locally finite combinatorial inverse monoid  $S$ , we are considering three rules for determining  $\mu(s, e)$ :

(1) The collection of quotients in  $C_F(S)$  of an object  $e$  forms a locally finite partially ordered set in a natural way. The first rule uses the Möbius function of this locally finite partially ordered set. For any  $e \in \text{Ob}C_F(S)$ , the set  $Q(e)$  of all quotient objects of  $e$  with the canonical partial order  $\preceq$  is a locally finite lattice, and (see [9, Theorem 3.5 (i)]):

$$\mu(s, e) = \mu_{Q(e)}((s, e), (e, e)),$$

for any morphism  $(s, e)$  of  $C_F(S)$ , where  $\mu_{Q(e)}$  is the Möbius function of  $(Q(e), \preceq)$ .

(2) The natural partial order  $\leq$  on  $S$ , restricted to idempotents is given by:  $e, f \in E(S) : e \leq f \Leftrightarrow e = ef = fe$ . The second rule for determining the value of the function  $\mu$  at  $(s, e)$  uses the Möbius function  $\mu_{E(eSe)}$  of the locally finite lattice  $(E(eSe), \leq)$ . We have:

$$\mu(s, e) = \mu_{E(eSe)}(s^{-1}s),$$

for any morphism  $(s, e)$  of  $C_F(S)$  (see [9, Theorem 3.5 (ii)]).

(3) The third rule uses Lawvere intervals. If  $f$  is a morphism of a small category  $C$  then the Lawvere interval  $I(f)$  of  $f$  is a category with the set of factorizations of  $f$  as objects. The morphisms of  $I(f)$  are morphism of  $C$  which are compatible with factorizations of  $f$ . That is, if  $f = uv$  and  $f = u'v'$  in  $C$  then  $h : \text{Dom}g(= \text{Codom}h) \rightarrow \text{Dom}g'(= \text{Codom}h')$  is a morphism in  $I(f)$  from  $uv$  to  $u'v'$  if  $uh = u'$  and  $hv' = v$ . The composition in  $I(f)$  is the same as in  $C$ .

**Theorem 2.1.** ([2]) *A small category  $C$  is Möbius if and only if all Lawvere intervals of  $C$  are finite and one-way (i.e.  $\text{Hom}_C(X, Y) \neq \emptyset, \text{Hom}_C(Y, X) \neq \emptyset \Rightarrow X = Y$ ; and  $|\text{Hom}_C(X, X)| = 1$  for any object  $X$  in  $C$ ).*

**Theorem 2.2.** ([2],[12]) *If  $C$  is Möbius then any Lawvere interval is Möbius and for any morphism  $f : X \rightarrow Y$  in  $C$ ,*

$$\mu(f) = \mu_{I(f)}(f),$$

where  $\mu_{I(f)}$  is the Möbius function of  $I(f)$ . (The morphism  $f$  of  $C$  is also a morphism of  $I(f)$  from  $f1_X$  to  $1_Y f$ ; and if  $I(f)$  is a bounded poset then  $\mu(f) = \mu_{I(f)}(0, 1)$  with  $0 = f1_X$  and  $1 = 1_Y f$ .)

**Theorem 2.3.** ([12]) *If a small category  $C$  is Möbius that arises from a combinatorial inverse semigroup  $S$  then any Lawvere interval is a finite lattice (as a category), and for any morphism  $f$ ,*

$$\mu(f) = \mu_{I(f)}(0, 1),$$

where  $\mu_{I(f)}$  is the Möbius function of the finite lattice  $I(f)$ ,  $0$  is the least element and  $1$  is the greatest element of  $I(f)$ .

In the next section we illustrate the computation of the Möbius function with an example. We consider a category  $C_m$  and based on the above theorems:

- 1) we show that  $C_m$  is Möbius;
- 2) we find the Möbius function of  $C_m$ .

We focus our attention on the above problems and we will look at the starting inverse semigroup only at the end of the paper.

### 3. AN EXAMPLE

Let  $m$  be a positive integer ( $m > 1$ ),  $Z_m$  the cyclic group of addition modulo  $m$ ,  $Z_-$  the set of non-positive integers, and  $Z_+$  the set of non-negative integers. Now, let  $C_m$  be the category defined by:

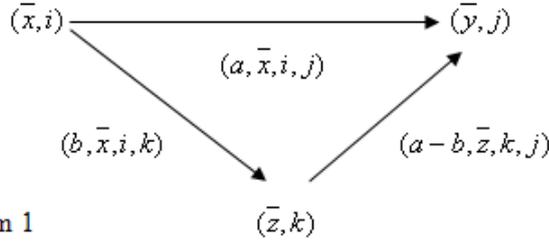
- $ObC_m = Z_m \times Z_-$
- $Hom_{C_m}((\bar{x}, i), (\bar{y}, j)) = \{(a, \bar{x}, i, j) | a \in Z_+, a \leq i - j, \bar{a} + \bar{x} = \bar{y}\}$
- $(b, \bar{y}, j, k) \circ (a, \bar{x}, i, j) = (a + b, \bar{x}, i, k)$  is the composition of two morphisms  $(a, \bar{x}, i, j) : (\bar{x}, i) \rightarrow (\bar{y}, j)$  and  $(b, \bar{y}, j, k) : (\bar{y}, j) \rightarrow (\bar{z}, k)$ .

The first question is this: is the above category Möbius? We will answer this question by applying Theorem 2.1. The second question concerns the computation of the Möbius function  $\mu$  of  $C_m$ .

Let  $(a, \bar{x}, i, j) : (\bar{x}, i) \rightarrow (\bar{y}, j)$  be a morphism in  $C_m$ . First, we examine the factorizations in  $C_m$  of this morphism, that is the objects of the category (of the Lawvere interval)  $I(a, \bar{x}, i, j)$ . Let

$$(a, \bar{x}, i, j) = (a - b, \bar{z}, k, j) \circ (b, \bar{x}, i, k)$$

be a factorization of  $(a, \bar{x}, i, j)$ , i.e. the Diagram 1 is commutative.



**Diagram 1**

We have:

$$a > 0, i, j \leq 0, a \leq i - j \text{ and } \bar{a} + \bar{x} = \bar{y} \text{ (since } (a, \bar{x}, i, j) \in Hom_{C_m}(\bar{x}, i), (\bar{y}, j)\text{)}$$

and

$$0 \leq b \leq a, k \leq 0, b \leq i - k, a - b \leq k - j, \bar{b} + \bar{x} = \bar{z} \text{ (} \bar{a} - \bar{b} + \bar{z} = \bar{y} \text{ is a consequence).}$$

Now, for a fixed integer  $b$ ,  $0 \leq b \leq a$ ,

- $\bar{z}$  is uniquely determined by:  $\bar{z} = \bar{b} + \bar{x}$ . We denote by  $\bar{z}_b$  this residue class.

- The values of  $k$  are determined by the condition:  $a - b + j \leq k \leq i - b$ .  
Since,

$$i - b - (a - b + j) = i - j - a \geq 0,$$

it follows that the values of  $k$  are the following:

$$k_0 = a - b + j, \quad k_1 = a - b + j + 1, \dots, k_t = a - b + j + t, \dots$$

$$\dots, k_{i-j-a} = i - b.$$

Thus,

**Proposition 3.1.** *The set of objects of  $I(a, \bar{x}, i, j)$  is finite for any morphism  $(a, \bar{x}, i, j)$  of  $C_m$ .*

With the above notations there exists a morphism of  $C_m$  from  $(\bar{z}_b, k_t)$  to  $(\bar{z}_b, k_p)$  such that the Diagram 2 is commutative if and only if  $k_p \leq k_t$  (that is, if and only if  $p \leq t$ ), where  $0 \leq p, t \leq i - j - a$ . It is clear that this morphism  $(0, \bar{z}_b, k_t, k_p)$  of  $C_m$  is uniquely determined by the commutative Diagram 2.

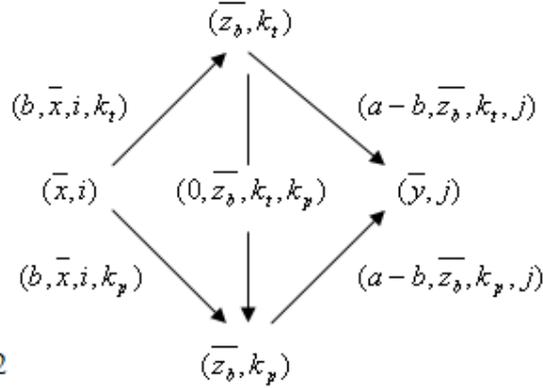


Diagram 2

Thus,

**Proposition 3.2.** *The set of morphisms  $Hom_{I(a, \bar{x}, i, j)}(X_{b,t}, X_{b,p})$ , where*

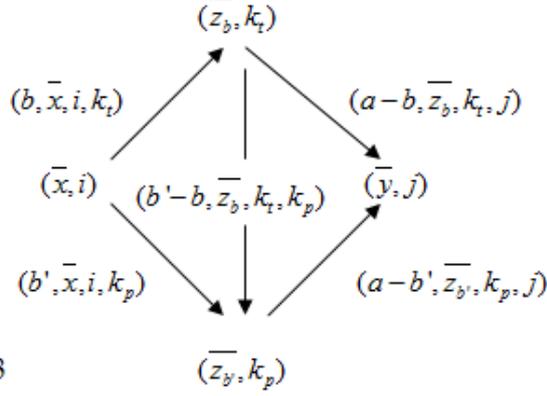
$$X_{b,t} = (\bar{x}, i) \xrightarrow{(b, \bar{x}, i, k_t)} (\bar{z}_b, k_t) \xrightarrow{(a-b, \bar{z}_b, k_t, j)} (\bar{y}, j)$$

and

$$X_{b,p} = (\bar{x}, i) \xrightarrow{(b, \bar{x}, i, k_p)} (\bar{z}_b, k_p) \xrightarrow{(a-b, \bar{z}_b, k_p, j)} (\bar{y}, j)$$

is non-empty and it is a singleton if and only if  $p \leq t$ .

A similar examination of the commutativity in  $C_m$  of the Diagram 3 leads us to conclude that (the notations are the same as in Proposition 3.2):



**Proposition 3.3.** *The set of morphisms  $\text{Hom}_{I(a, \bar{x}, i, j)}(X_{b, t}, X_{b', p})$  is non-empty and it is a singleton if and only if  $b \leq b'$  and  $p \leq t$ .*

By Theorem 2.1 and the above results, we have:

**Proposition 3.4.** *The category  $C_m$  is a Möbius category, and every Lawvere interval of  $C_m$  is a finite lattice.*

The Diagram 4 is the Hasse diagram of the lattice  $I(a, \bar{x}, i, j)$ .

Now, a routine evaluation on the Hasse Diagram 4 of the poset's Möbius function  $\mu_{I(a, \bar{x}, i, j)}$ , and the equalities  $\mu(a, \bar{x}, i, j) = \mu_{I(a, \bar{x}, i, j)}(a, \bar{x}, i, j) = \mu_{I(a, \bar{x}, i, j)}(0, 1)$  implies:

**Proposition 3.5.** *The Möbius function  $\mu$  of the Möbius category  $C_m$  is given by:*

$$\mu(a, \bar{x}, i, j) = \begin{cases} 1 & \text{if } a = 0 \text{ and } j = i \quad \text{or } a = 1 \text{ and } j = i - 2 \\ -1 & \text{if } a = 0 \text{ and } j = i - 1 \quad \text{or } a = 1 \text{ and } j = i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

#### 4. FINAL REMARKS

The Theorem 2.3 suggests that the Möbius category  $C_m$  arises from a combinatorial inverse semigroup. An examination of the Möbius function leads us to the free monogenic inverse monoid (see [10], Proposition 3.3). But  $C_m$  is not the Möbius category (i.e. the reduced division category) of the free monogenic inverse monoid. The category  $C_m$  is the Möbius category of a  $\varrho$ -semigroup of the free monogenic inverse monoid, namely of the one-dimensional tiling semigroup in the periodic case: the period has length  $m$  and involves each tile exactly once (see [13]). By a change from a combinatorial inverse monoid to his  $\varrho$ -semigroup, the Möbius function becomes an invariant ([13, Theorem 2.4]).

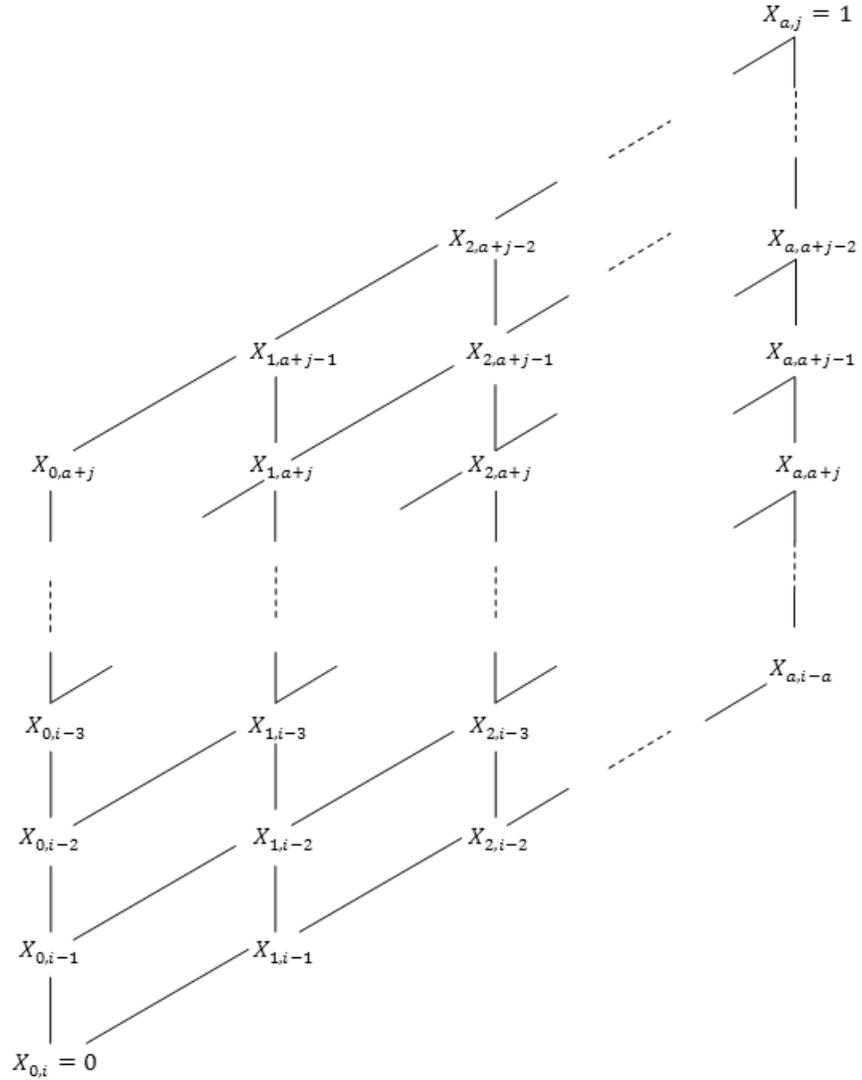


Diagram 4  
 (b=0)                      (b=1)                      (b=2)                      -----                      (b=a)

Now, another way to compute the Möbius function of a Möbius category is given below (using a comparison method). Let  $D_m$  be the Möbius category defined by:

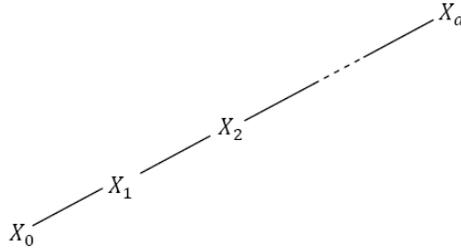
- $ObD_m = Z_m$
- $Hom_{D_m}(\bar{x}, \bar{y}) = \{(\alpha, \bar{x}) \in Z_+ \times Z_m \mid \alpha \geq x \text{ and } \alpha \equiv y \pmod{m}\}$ ;
- If  $(\alpha, \bar{x}) \in Hom_{D_m}(\bar{x}, \bar{y})$  and  $(\beta, \bar{y}) \in Hom_{D_m}(\bar{y}, \bar{z})$  then  $(\beta, \bar{y}) \cdot (\alpha, \bar{x}) = (\beta - y + \alpha, \bar{x})$ .

The functor  $F : C_m \rightarrow D_m$  given by:

- $F(\bar{x}, i) = \bar{x}$
- $F(a, \bar{x}, i, j) = (a + x, \bar{x})$

is surjective that commute with Lawvere intervals  $I$ . The chain of Diagram 5 is the image through the functor  $F$  of the Lawvere interval  $I(a, \bar{x}, i, j)$  (Diagram 4).

Diagram 5



It follows that (where  $\alpha = a + x$ ):

$$\mu(\alpha, \bar{x}) = \begin{cases} 1 & \text{if } \alpha = x \\ -1 & \text{if } \alpha = x + 1 \\ 0 & \text{if } \alpha \geq x + 2 \end{cases}$$

$\mu$  being the Möbius function of the Möbius category  $D_m$ .

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