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Rational Function Series

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Our problem is to describe the *domain of convergence* of

$$\sum_{k=1}^{\infty} r_{n_k}(z) \tag{1}$$

where $\{n_k\}_k$ is a sequence of strictly increasing positive integers, and r_n is a rational function of *degree* n with complex coefficients. The degree of a rational function is the number of its poles (or zeros) on $\overline{\mathbb{C}}$ (the extended complex plane). By "domain of convergence" D we mean the union of all open sets on $\overline{\mathbb{C}}$ (the one-point compactification of the complex plane) with the property that the series converges locally uniformly in D. We note that D does not contain any pole, and that the limit is continuous in D according to the uniform limit theorem.

The Weierstrass M-test (or Abel's uniform convergence test) provides a general criterion for uniform convergence on a given set, but not a description of that set based on the asymptotic behaviors of r_n. But of course in the best known case, the case of power series, we know a precise description of D based on the coefficients. And, in the most general case of polynomial series Götz [1] has described D in terms of the asymptotic behaviors of the leading coefficients and the zeros. The present work grew out of [2], and its technique is a natural evolution of what used by Walsh [3].

Let $\{y_k\}_k$ be a convergent sequence of points on $\overline{\mathbb{C}}$, say $y_k \to y_\infty$, such that

$$\limsup_{k\to\infty} |r_{n_k}(y_k)|^{\frac{1}{n_k}} = 1.$$

Let ν_k be a unit signed measure on $\overline{\mathbb{C}}$ with mass $+1/n_k$ at each pole and $-1/n_k$ at each zero of r_{n_k}. We assume that ν_k converges to a signed measure ν_∞ in the weak* topology of the space of all finite signed measures on Borel subsets of $\overline{\mathbb{C}}$, which means that

$$\int f(t)d\nu_k(t) \to \int f(t)d\nu_\infty(t)$$

for all f continuous real-valued functions on $\overline{\mathbb{C}}$. The Jordan decomposition of ν_∞ can be written as

$$\nu_\infty = \nu_\infty^+ - \nu_\infty^-.$$

We assume that $Supp(\nu^+_\infty) \cap Supp(\nu^-_\infty) = \varnothing$. For each such a signed measure ν with $\nu^+(1) = \nu^-(1) = 1$, we define a bivariate potential

$$w^\nu(x, y) := \int \log \frac{|y - t|}{|x - t|} d\nu(t)$$

for $x \notin Supp(\nu^+)$ and $y \notin Supp(\nu^-)$. Then the rational function series (1) converges locally uniformly in

$$E^- := \{x : w^{\nu_\infty}(x, y_\infty) < 0\}$$

and diverges q.a.e. in

$$E^- := \{x : w^{\nu_\infty}(x, y_\infty) > 0\}.$$

References

[1] Götz M. (2005)
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[3] Walsh J. L. (1935, 1969)
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