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OBSTRUCTION TO DEFORMATIONS OF PAIRS OF RATIONAL CURVES AND HYPERSURFACES

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Abstract. In this paper, we prove that if the first order deformation of the pair of a hypersurface $f_0 \subset \mathbf{P}^n$ and an imbedded rational curve $C_0 \subset div(f_0) \subset \mathbf{P}^n$ exists in some directions, then

$$H^1(N_{C_0/div(f_0)}(1)) = 0,$$

where $N_{C_0/div(f_0)}(1) = N_{C_0/div(f_0)} \otimes \mathcal{O}_{\mathbf{P}^n}(1)|_{C_0}$.

1. Introduction. We work over complex numbers. Let $f_0 \in H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ such that the hypersurface $X_0 = div(f_0)$ in $\mathbf{P}^n$ is smooth. Let $C_0 \subset X$ be a smooth rational curve. Let

$$\mathbb{H}^1(T_{X_0} \to N_{C_0/X_0})$$

be the hypercohomology of the complex, that is isomorphic to the tangent space of the deformations of the pair $C_0 \subset X_0$, and $H^1(T_{X_0})$, the space that is isomorphic to the tangent space of the moduli space of hypersurfaces at the point X_0. There is a known diagram:

$$\begin{array}{ccc} & & \mathbb{H}^1(T_{X_0} \to N_{C_0/X_0}) \\ & & \downarrow\phi \\ T_{[f_0]}\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) & \overset{\psi}{\twoheadrightarrow} & H^1(T_{X_0}) \end{array} \tag{1.1}$$

where the map ψ is surjective, and $[f_0]$ denotes the point in $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ that represents the smooth hypersurface X_0.

Let $L_i \in H^0(\mathcal{O}_{\mathbf{P}^n}(1))$, $i = 0, \cdots, h$ be some non-zero sections (no need to be generic) that satisfy

$$\{L_i = 0\} \cap \{L_j = 0\} \cap C_0 = \emptyset, i \neq j. \tag{1.2}$$

We consider the family

$$\mathcal{X} \subset \mathbf{P}^n \times A$$

of hypersurfaces of degree h in the form

$$\{F(a_1, \cdots, a_h, x) = f_0(x) + \sum_{i=0}^{h} a_i L_0(x) \cdots \hat{L}_i(x) \cdots L_h(x) = 0\}, \quad (omit\ L_i)$$

where A is the open set of $\mathbf{C}^{h+1} = \{(a_0, \cdots, a_h)\}$ that parametrizes the smooth hypersurfaces in the family $\mathcal{X}$. So $A \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ is the parameter space of $\mathcal{X}$.

Theorem 1.1.
(1) Let $T_{[f_0]}A = A$ under the affine structure of A. If

$$\psi(A) \subset image(\phi), \tag{1.3}$$

then

$$H^1(N_{C_0/X_0} \otimes \mathcal{O}_{\mathbf{P}^n}(1)|_{C_0}) = 0. \tag{1.4}$$

2. First order deformations of the pair. In this section, we give another description of the condition (1.3), which will be used throughout. Let

$$S \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$$

be an irreducible subvariety that contains $[f_0]$ and is smooth at $[f_0]$. Let

$$\mathcal{X}_S \subset \mathbf{P}^n \times S, \tag{2.1}$$

$$\mathcal{X}_S = \{(x, [f]) : [f] \in S, f(x) = 0\}. \tag{2.2}$$

be the universal hypersurface for $S \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$.

Let

$$\begin{aligned} c_0 : &\mathbf{P}^1 \to X_0 \\ &t \to c_0(t) \end{aligned}$$

be the imbedding from $\mathbf{P}^1$ onto C_0. Let

$$\begin{aligned} \bar{c}_0 : &\mathbf{P}^1 \to C_0 \times \{[f_0]\} \subset \mathcal{X}_S \\ &t \to (c_0(t), [f_0]) \end{aligned}$$

be the imbedding determined by c_0. The projection

$$P_S : \mathcal{X}_S \to S$$

has a differential map

$$T_{(q,[f_0])}\mathcal{X}_S \quad \to \quad T_{[f_0]}S, \quad q \in C_0$$

which can be extended to a bundle map

$$(P_S)_* : \bar{c}_0^*(T_{\mathcal{X}_S}) \quad \to \quad T_{[f_0]}S \otimes \mathcal{O}_{\mathbf{P}^1}.$$

At last we obtain a homomorphism on the vector spaces

$$P_S^s : H^0(\bar{c}_0^*(T_{\mathcal{X}_S})) \to T_{[f_0]}S, \tag{2.3}$$

where $T_{[f_0]}S \simeq H^0(T_{[f_0]}S \otimes \mathcal{O}_{\mathbf{P}^1})$ is the space of global sections of the trivial bundle whose each fibre is $T_{[f_0]}S$.

LEMMA 2.1.

$$\psi(T_{[f_0]}S) \subset image(\phi)$$

if and only if P^s_S *is surjective.*

Proof. Let M_d be the parameter space of imbeddings $\mathbf{P}^1 \to \mathbf{P}^n$, whose image has degree d. So M_d is an open set of

$$\mathbf{P}(\oplus_{n+1}\mathcal{O}_{\mathbf{P}^n}(d)).$$

The map c_0 represents a point in M_d which is still denoted by c_0. Let $\mathcal{X}_n$ be the universal hypersurface for $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$. (defined in formula (2.2)). Let

$$\begin{aligned} &\Gamma \subset M_d \times \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) \\ &\Gamma = \{(c,[f]) : c^*(f) = 0\}. \end{aligned} \tag{2.4}$$

be the incidence scheme containing the point $(c_0, [f_0])$. Let $T_{(c_0,[f_0])}\Gamma$ be the Zariski tangent space of Γ. Let e be the evaluation map

$$\begin{aligned} e: \quad &\Gamma \times \mathbf{P}^1 \to \mathcal{X}_n \\ &(c,[f],t) \to (c(t),[f]). \end{aligned}$$

Its differential map induces a bundle map

$$e_*: \quad T_{(c_0,[f_0])}\Gamma \otimes \mathcal{O}_{\mathbf{P}^1} \to c_0^*(T_{\mathcal{X}_n}),$$

It further induces a homomorphism on the cohomologies:

$$e^s: \quad T_{(c_0,[f_0])}\Gamma \to H^0(c_0^*(T_{\mathcal{X}_n})),$$

where $T_{(c_0,[f_0])}\Gamma = H^0(T_{(c_0,[f_0])}\Gamma \otimes \mathcal{O}_{\mathbf{P}^1})$. Also there is a surjective map η:

$$T_{(C_0,[f_0])}\Gamma \twoheadrightarrow \mathbb{H}^1(T_{X_0} \to N_{C_0/X_0}),$$

such that the following diagram commutes

$$\begin{array}{ccccc} T_{(C_0,[f_0])}\Gamma & = & T_{(C_0,[f_0])}\Gamma & \stackrel{\eta}{\twoheadrightarrow} & \mathbb{H}^1(T_{X_0} \to N_{C_0/X_0}) \\ \downarrow e^s & & \downarrow & & \downarrow \phi \\ H^0(\bar{c}_0^*(T_{\mathcal{X}_n})) & \stackrel{P^s_\eta}{\to} & T_{[f_0]}\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h))) & \stackrel{\psi}{\twoheadrightarrow} & H^1(T_{X_0}), \end{array} \tag{2.5}$$

where P^s_n is the corresponding map in formula (2.3). Because

$$T_{c_0}M_d \to H^0(c_0^*(T_{\mathbf{P}^n}))$$

is surjective (it is an isomorphism), e^s has to be surjective. Then the lemma is true for $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$. Now we consider the subvariety $S \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ in the lemma. If $\psi(T_{[f_0]}S) \subset image(\phi)$, for any $\alpha \in T_{[f_0]}S$, we apply the diagram to find a section $\sigma \in H^0(\bar{c}_0^*(T_{\mathcal{X}_n}))$ such that $P_n^s(\sigma) = \alpha$. Because $P_n^s(\sigma) = \alpha \in T_{[f_0]}S$, σ must be in the subspace $H^0(\bar{c}_0^*(T_{\mathcal{X}_S}))$ of $H^0(\bar{c}_0^*(T_{\mathcal{X}_n}))$. Thus P_S^s is surjective. Conversely we suppose P_S^s is surjective. For any $\alpha \in T_{[f_0]}S$, using the commutative diagram, we obtain

$$\psi(\alpha) \in \phi \circ \eta \circ (e^s)^{-1} \circ (P_S^s)^{-1}(\alpha).$$

We complete the proof. □

3. The proof of theorem 1.1. The main idea of the proof is to replace the bundle $T_{\mathbf{P}^n}(1)|_{C_0}$ by some isomorphic bundle $(\frac{T_{\mathcal{X}}(1)}{G(1)})|_{C_0}$.

Recall the notation

$$F(a_1, \cdots, a_h, x) = f_0(x) + \sum_{i=0}^{h} a_i L_0(x) \cdots \hat{L}_i(x) \cdots L_h(x), \quad (omit\ L_i)$$

is the universal polynomial. Thus

$$\{F = 0\} = \mathcal{X} \subset \mathbf{P}^n \times A.$$

is the universal hypersurface, which is smooth. Let $W \subset \mathbf{P}^n$ denote the complement of the proper subvariety

$$\cup_{h \geq j > i \geq 0} \{L_i = L_j = 0\}.$$

Let

$$\mathcal{X}_W = \mathcal{X} \cap (W \times A). \tag{3.1}$$

Let

$$u_i = L_0 \frac{\partial}{\partial a_0} - L_i \frac{\partial}{\partial a_i}, i = 1, \cdots, h \tag{3.2}$$

be sections of $\mathcal{O}_{\mathbf{P}^n}(1) \otimes T_A$. Since u_i annihilate F, they are tangent to $\mathcal{X}_W$. So let

$$G(1) \subset T_{\mathcal{X}_W}(1) \tag{3.3}$$

be the vector bundle of rank h over $\mathcal{X}_W$ that is generated by the sections u_i.

Lemma 3.1.

$$\frac{T_{\mathcal{X}_W}(1)}{G(1)} \simeq T_{(W \times A)/A}(1)|_{\mathcal{X}_W}, \tag{3.4}$$

where $T_{(W \times A)/A}(1) = (T_W(1) \oplus \{0\})$ *is the twisted relative tangent bundle of the projection* $W \times A \to A$.

Remark This theorem does not require any assumptions. This is a fact about this family of hypersurfaces.

Proof. Consider the exact sequence

$$0 \to \frac{T_{\mathcal{X}_W}(1)}{G(1)} \to \frac{T_{(W\times A)}(1)}{G(1)} \to \mathcal{D} \to 0. \tag{3.5}$$

of bundles over $\mathcal{X}_W$, where $\mathcal{D}$ is some quotient bundle over $\mathcal{X}_W$. Easy to see

$$c_1(\mathcal{D}) = c_1(\mathcal{O}_{\mathbf{P}^n}(h+1))|_{\mathcal{X}_W}. \tag{3.6}$$

Let s be a generic section of $\mathcal{O}_{\mathbf{P}^n}(1)$. Let σ be the reduction of $s\frac{\partial}{\partial a_0}$ in $\frac{T_{(W\times A)}(1)}{G(1)}$. Notice the zeros of σ is exactly

$$div(\sigma) = div(sL_1\cdots L_h). \tag{3.7}$$

Since $sL_1\cdots L_h \in H^0(\mathcal{O}_{\mathbf{P}^n}(h+1))$, σ splits the sequence (3.5). If $L_s \subset \frac{T_{(W\times A)}(1)}{G(1)}$ is the line bundle generated by σ,

$$L_s \oplus \frac{T_{\mathcal{X}_W}(1)}{G(1)} = \frac{T_{(W\times A)}(1)}{G(1)}, \tag{3.8}$$

as bundles over $\mathcal{X}_W$. Secondly, we have another exact sequence

$$0 \to T_{(W\times A)/A}(1) \to \frac{T_{(W\times A)}(1)}{G(1)} \to \mathcal{D}' \to 0. \tag{3.9}$$

of bundles over $\mathcal{X}_W$, where $\mathcal{D}'$ is some quotient bundle over $\mathcal{X}_W$. By the direct calculation (note $G(1)$ is a trivial bundle):

$$c_1(\mathcal{D}') = c_1(T_{W\times A/W}(1)) = (h+1)(c_1(\mathcal{O}_{\mathbf{P}^n}(1)))|_{\mathcal{X}_W}$$

As above, σ splits this sequence (3.9). Hence

$$L_s \oplus T_{(W\times A)/A}(1) = \frac{T_{(W\times A)}(1)}{G(1)}. \tag{3.10}$$

Comparing (3.8), (3.10), we obtain

$$\frac{T_{\mathcal{X}_W}(1)}{G(1)} \simeq T_{(W\times A)/A}(1), \tag{3.11}$$

over $\mathcal{X}_W$. □

Proof. of theorem 1.1:

By lemma 2.1, because ϕ is onto $\psi(A)$, P_A^s is onto $T_{f_0}A = A$. This gives us a natural bundle decomposition of $\bar{c}_0^*(T_{\mathcal{X}}(1))$ in the following way: Note $\bar{c}_0^*(T_{\mathcal{X}}(1))$ has sections $(P_A^s)^{-1}(\frac{\partial}{\partial a_j}), j = 0, \cdots, h$. Let $\sigma_i \in (P_A^s)^{-1}(\frac{\partial}{\partial a_j})$, $i =$

$0, \cdots, h$ be a vector in each inverse $(P^s_A)^{-1}(\frac{\partial}{\partial a_j})$. Then all sections $\{\sigma_j\}_j$ generate a trivial subbundle $\mathcal{E}$

$$\mathcal{E} = \oplus_{h+1} \mathcal{O}_{\mathbf{P}^1}.$$

This subbundle gives a decomposition

$$\bar{c}_0^*(T_{\mathcal{X}}) \simeq \mathcal{E} \oplus \bar{c}_0^*(T_{\mathcal{X}/A}), \tag{3.12}$$

Tensoring it with $\bar{c}_0^*(\mathcal{O}_{\mathbf{P}^n}(1))$, we obtain

$$\bar{c}_0^*(T_{\mathcal{X}}(1)) \simeq \mathcal{E}(1) \oplus \bar{c}_0^*(T_{\mathcal{X}/A}(1)). \tag{3.13}$$

Notice the relative tangent bundle is

$$c_0^*(T_{\mathcal{X}/A}(1)) = c_0^*(T_{X_0}(1)).$$

Now consider the exact sequence from lemma 3.1:

$$0 \rightarrow \bar{c}_0^*(G(1)) \rightarrow \bar{c}_0^*(T_{\mathcal{X}}(1)) \rightarrow c_0^*(T_{\mathbf{P}^n}(1)) \rightarrow 0. \tag{3.14}$$

We obtain the exact sequence on the chomology groups

$$H^1(\bar{c}_0^*(G(1))) \rightarrow H^1(\bar{c}_0^*(T_{\mathcal{X}}(1))) \rightarrow H^1(c_0^*(T_{\mathbf{P}^n}(1))). \tag{3.15}$$

Because $c_0^*(T_{\mathbf{P}^n}(1))$ is generated by global sections,

$$H^1(c_0^*(T_{\mathbf{P}^n}(1))) = 0.$$

Clearly the trivial bundle $\bar{c}_0^*(G(1))$ should also have $H^1(\bar{c}_0^*(G(1))) = 0$. Then (3.15) implies

$$H^1(\bar{c}_0^*(T_{\mathcal{X}}(1))) = 0.$$

Using the decomposition in (3.13),

$$H^1(c_0^*(T_{X_0}(1))) = H^1(\bar{c}_0^*(T_{\mathcal{X}}(1))) = 0.$$

Thus

$$H^1(N_{C_0/X_0}(1)) = H^1(\frac{T_{X_0}(1)}{T_{C_0}(1)}) = 0.$$

This completes the proof.
□

4. Examples. In this section, we give 3 examples with applications of Theorem 1.1.

Example 4.1 Let $n = 4, h = 5$. We consider f_0 to be the Fermat quintic and C_0 a line on X_0.

$$H^1(N_{C_0/X_0}(1)) = H^0(\mathcal{O}_{\mathbf{P}^1}(0) \oplus \mathcal{O}_{\mathbf{P}^1}(-4)) \neq 0.$$

Our theorem 1.1 says that these values form an obstruction to the deformations of C_0 to other quintics. Indeed none of lines in X_0 can deform to all quintics in A. In [1], one can find detailed deformations of the pair $line \subset Fermat\ quintic$ without specifying A and the condition 1.3.

Example 4.2 Let $n = 4, h = 5$. Let

$$f_0 = lg_1 + qg_2$$

where l is linear and q is quadratic. Assume all l, q, g_i are generic. Let C_0 be a smooth rational curve of degree d, lying in the quadratic surface $\{l = q = 0\}$. Now $X_0 = div(f_0)$ is not smooth, but there are only 24 singular points. We may assume X_0 is smooth along C_0. Then theorem 1.1 should still be valid for such $C_0 \subset f_0$. Apply it to the pair (C_0, f_0). We found

$$H^1(N_{C_0/X_0}(1)) \simeq H^0(\mathcal{O}_{\mathbf{P}^1}(-3d) \oplus \mathcal{O}_{\mathbf{P}^1}(d-2)),$$

which is non-zero if $d \neq 1$. Thus if C_0 is not a line, then the pair $C_0 \subset X_0$ can't deform to all hypersurfaces in A' to the first order. In particular, they are obstructed to deform to all hypersurfaces to the first order.

Example 4.3 Let $n = 4, h = 5$. In this case, if C_0 can deform to all the hypersurfaces in A', by theorem 1.1, $H^1(N_{C_0/X_0}(1)) = 0$. Using the adjunction formula, it is easy to see

$$N_{C_0/X_0}(1) \simeq \mathcal{O}_{\mathbf{P}^1}(k+d) \oplus O_{\mathbf{P}^1}(-2-k+d) \tag{4.1}$$

where $k \geq -1$. Then $H^1(N_{C_0/X_0}(1)) \simeq H^0(\mathcal{O}_{\mathbf{P}^1}(-k-d-2) \oplus O_{\mathbf{P}^1}(k-d)) = 0$ implies an upper bound of k, i.e., $k < d$.

In this case, we also have the Clemens' conjecture [2] that is equivalent to the assertion that for a generic quintic X_0, Kodaira's general sufficient condition ([4])

$$H^1(N_{C_0/X_0}) = 0$$

(without a twist) is also a necessary condition for C_0 to deform to all quintics. Our theorem 1.1 did not prove the Clemens' conjecture because of the twist on the normal bundle N_{C_0/X_0}. Instead, we obtain an upper bound of k above.

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