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CANTORVALS, THEIR TOPOLOGICAL STRUCTURE, AND THE GENERALIZATION OF THE GUTHRIE/NYMAN SET

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Synopsis:

With interesting topological properties, the Cantor set is worth studying for itself. In other areas, topological structures arise that are in fact homeomorphic to the Cantor set. In particular, we see sets that are homeomorphic to the Cantor set which result from the subsums of particular series, as well as linear combinations of algebraic sums of Cantor sets. These also result in what has been termed a Cantorval, which we also investigate.

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Abstract. With interesting topological properties, the Cantor set is worth studying for itself. In other areas, topological structures arise that are in fact homeomorphic to the Cantor set. In particular, we see sets that are homeomorphic to the Cantor set which result from the subsums of particular series, as well as linear combinations of algebraic sums of Cantor sets. These also result in what has been termed a Cantorval, which we also investigate.

In 1914 the following results were discovered by Kakeya:
Let

$$R_n = \sum_{i=n+1}^{\infty} x_i$$

The subsum set $\sum \{x_i\}_{i=1}^{\infty}$ is either a perfect set, the finite union of closed intervals if and only $a_n \leq R_n$ for all n for n sufficiently large, or

homeomorphic to the Cantor set if $a_n > R_n$ for n sufficiently large.

He also conjectured $\sum \{x_i\}_{i=1}^{\infty}$ is nowhere dense, and as a result homeomorphic to the Cantor set, if $a_n > R_n$ for infinitely many n.

The first counterexample to this conjecture was given by Weinstein and Shapiro without proof in 1980, namely

$$\sum_{i=1}^{\infty} \left[\varepsilon_{5i-4}\left(\frac{8}{10^i}\right) + \varepsilon_{5i-3}\left(\frac{7}{10^i}\right) + \varepsilon_{5i-2}\left(\frac{6}{10^i}\right) + \varepsilon_{2i-1}\left(\frac{5}{10^i}\right) + \varepsilon_{5i}\left(\frac{4}{10^i}\right) + \right], \ \varepsilon_i \in \{0,1\}, \ \forall\, i \in \mathbb{N}.$$

Shortly after in 1984, Ferens gave another example which included proof, namely,

$$\sum_{i=1}^{\infty} \left[\varepsilon_{5i-4}\left(\frac{7}{27^i}\right) + \varepsilon_{5i-3}\left(\frac{6}{27^i}\right) + \varepsilon_{5i-2}\left(\frac{5}{27^i}\right) + \varepsilon_{2i-1}\left(\frac{4}{27^i}\right) + \varepsilon_{5i}\left(\frac{3}{27^i}\right) + \right], \ \varepsilon_i \in \{0,1\}, \ \forall\, i \in \mathbb{N}.$$

However, not long after in 1988, Guthrie and Nymann provided a simple counter example of a subsum set that is not the finite union of closed intervals yet has non-empty interior, namely,

$$T = \sum_{i=1}^{\infty} \left[\varepsilon_{2i-1}\left(\frac{3}{4^i}\right) + \varepsilon_{2i}\left(\frac{2}{4^i}\right)\right], \ \varepsilon_i \in \{0,1\}, \ \forall\, i \in \mathbb{N} \qquad (0.1)$$

and went on to apply it to described the topological structure of the set of subsums $\sum \{x_i\}_{i=1}^{\infty}$.

Here we generalize the Guthrie/Nymann set(0.1).

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Part 1. **Cantorvals**

1. Early Work

Following the notation of Nitecki[7] we define the subsum set of a null sequence.

Definition 1. Let where $\xi = \xi_1\xi_2\cdots$, then the **subsum set** of a null sequence $\{x_n\}$, where $x_n \to 0$ as $n \to \infty$ is the collection

$$\sum \{x_i\}_{i=1}^{\infty} \tag{1.1}$$

of all numbers of the form

$$x(\varepsilon) = \sum_{i=1}^{\infty} \varepsilon_i x_i,\, \varepsilon_i \in \{0,1\}, \forall\, i \in \mathbb{N} \tag{1.2}$$

for which the series converges.

Definition 2. Let

$$R_n = \sum_{i=n+1}^{\infty} x_i$$

denote the n-th "tail" of 1.2

In 1914 the following results were discovered by Kakeya[4].

Theorem 3. *(Kakeya) The subsum set* $\sum \{x_i\}_{i=1}^{\infty}$ *is,*

(1) a perfect set,
(2) the finite union of closed intervals if and only $a_n \leq R_n$ for all n, or for n sufficiently large, or
(3) homeomorphic to the Cantor set if $a_n > R_n$ for n sufficiently large.

He also conjectured $\sum \{x_i\}_{i=1}^{\infty}$ is nowhere dense, and as a result homeomorphic to the Cantor set, if $a_n > R_n$ for infinitely many n.

The first counterexample to this conjecture was given by Weinstein and Shapiro[**?**] without proof in 1980, namely

$$\sum_{i=1}^{\infty}\left[\varepsilon_{5i-4}\left(\frac{8}{10^i}\right)+\varepsilon_{5i-3}\left(\frac{7}{10^i}\right)+\varepsilon_{5i-2}\left(\frac{6}{10^i}\right)+\varepsilon_{2i-1}\left(\frac{5}{10^i}\right)+\varepsilon_{5i}\left(\frac{4}{10^i}\right)+\right],\ \varepsilon_i \in \{0,1\},\ \forall i \in \mathbb{N}.$$

Shortly after in 1984, Ferens[2] gave another example which included proof, namely,

$$\sum_{i=1}^{\infty}\left[\varepsilon_{5i-4}\left(\frac{7}{27^i}\right)+\varepsilon_{5i-3}\left(\frac{6}{27^i}\right)+\varepsilon_{5i-2}\left(\frac{5}{27^i}\right)+\varepsilon_{2i-1}\left(\frac{4}{27^i}\right)+\varepsilon_{5i}\left(\frac{3}{27^i}\right)+\right],\ \varepsilon_i \in \{0,1\},\ \forall i \in \mathbb{N}.$$

However, not long after in 1988, Guthrie and Nymann [3] provided a simple counter example of a subsum set that is not the finite union of closed intervals yet has non-empty interior, namely,

$$T = \sum_{i=1}^{\infty}\left[\varepsilon_{2i-1}\left(\frac{3}{4^i}\right)+\varepsilon_{2i}\left(\frac{2}{4^i}\right)\right],\ \varepsilon_i \in \{0,1\},\ \forall\, i \in \mathbb{N} \tag{1.3}$$

and went on to described its topological structure of the set of subsums $\sum \{x_i\}_{i=1}^{\infty}$ as follows:

Theorem 4. *(Guthrie-Nymann) If $\sum\{x_i\}_{i=1}^{\infty}$ is the set of subsums of a positive term convergent series $\sum_{i=1}^{\infty} x_i$, then $\sum\{x_i\}_{i=1}^{\infty}$ is one of the following:*

(1) A finite union of closed intervals,
(2) homeomorphic to the Cantor set, or
(3) homeomorphic to the set of all subsums of T (1.3)

2. Current Work

In 1994 Mendez and Oliveira also studied sets of the type described in condition (3) of theorem 5, and called them Cantorvals, thus we can define a Cantorval as follows:

Definition 5. A **symmetric Cantorval** is a nonempty compact set $S \subset \mathbb{R}$ such that,

(1) $S = \overline{S^{\circ}}$
(2) The endpoints of any nontrivial component of S are accumulation points of trivial components of S.

and as a result the theorem of Guthrie and Nyman [3] can be rewritten as follows

Theorem 6. *(Guthrie-Nymann) If $0 < x_{i+1} \leq x_i$, and*

$$x(\varepsilon) = \sum_{i=1}^{\infty} \varepsilon_i x_i,\, \varepsilon_i \in \{0,1\}, \forall\, i \in \mathbb{N} \tag{2.1}$$

then its subsum set is either,

(1) a Cantor set,
(2) the finite union of disjoint closed intervals, or
(3) a symmetric Cantorval

Proof. (Due to Guthrie[3]) Suppose that E is neither a finite union of intervals nor homeomorphic to the Cantor set. Then it is clear the complement of E must contain infinitely many intervals. E must contain infinitely many intervals as well, for there were only finitely many, then either $E \cap [0, \varepsilon)$ is an interval for some $\varepsilon > 0$ or $E \cap [0, \varepsilon)$ contains no interval for some $\varepsilon > 0$. If the former is true, the there is some tail of $\sum a_a$ which has an interval as its set of subsums, and therefore E would be a finite union of intervals. If the latter holds, then $E \cap [0, \varepsilon]$ is homeomorphic to the Cantor set, and there is a tail of $\sum a_n$ which has the Cantor set as its set of subsums. Thus E would be homeomorphic to the Cantor set. This is again a contradiction to our initial supposition. Thus E contains infinitely many intervals.

In fact, $E \cap [a, b]$ cannot be homeomorphic to the Cantor set for any $a, b \in E$, since every tail of $\sum a_n$ must have intervals in in its set of subsums. Suppose then that, for some $x \in E$,

$$E \cap (x, x + \varepsilon) = \emptyset \quad \text{for some } \varepsilon > 0$$

Then, since E is perfect, $E \cap (x - \varepsilon, x) \neq \emptyset$ for every $\varepsilon > 0$, and therefore there are intervals in E arbitrarily close to x.

We now define a strictly increasing mapping f from the union of all intervals of T onto the union of all intervals in E . We can define the mapping inductively. Begin by mapping the longest interval in T in a strictly increasing way onto the longest interval in E . There can be at most finitely many intervals of the same length in either set, so we may choose the left-most interval in case no one interval is longest.

After the nth step, $2^n - 1$ intervals $[\alpha_j, \beta_j] \subset T\,(1j2^n - 1, \beta_j < \alpha_{j+1})$ will have been identified, in a strictly increasing way, with intervals $\left[\alpha'_j, \beta'_j\right] \subset E$. Now repeat the above process on each subset of T lying in $[\beta_j, \alpha_{j+1}]\,(j < 2^n - 1)$ or in $[0, \alpha_1]$ or in $[\beta_{2n-1}, 5/3]$. That is, map the longest interval in every such portion of T to the longest interval in E lying in $\left[\beta'_j, \alpha'_{j+1}\right]\,(j < 2^n - 1)$ or in $[0, \alpha'_1]$ or in $\left[\beta'_{2n-1}, \sum a_k\right]$, respectively.

When f is defined in this way, it is a strictly increasing mapping of the union of all intervals in T onto the union of all intervals in E . The property verified above that each point of E (and of T) is the limit of a sequence chosen from the intervals of the set allows us to extend m continuouly to all of T, and guarantees that the extension will be onto E. The extension will be strictly increasing and, therefore, one-to-one. Since T is compact, f is the desired homeomorphism. □

Theorem 7. *Any two symmetric Cantorvals are homeomorphic.*

Proof. Consider two Cantorvals $\mathfrak{C}$ and $\mathfrak{C}'$. Taking the longest and rightmost, or leftmost, component of each, there exists is a unique affine, order preserving homeomorphism between these components. By definition, there are other components of $\mathfrak{C}$ or $\mathfrak{C}'$, respectively, to the right and the left of the one chosen. Now, its complement is contained in two disjoint intervals, and the part of each Cantorval in each of these intervals is also a Cantorval. By repeated application of this method we can pair the longest nontrivial component of the chosen one in $\mathfrak{C}$ with the corresponding one in $\mathfrak{C}'$, and thus resulting in an order preserving correspondence, and homeomorphism between the nontrivial components of $\mathfrak{C}$ and $\mathfrak{C}'$, and therefore an order preserving continuous map from $\mathfrak{C}$ onto $\mathfrak{C}'$ resulting in a homeomorphism from all of $\mathfrak{C}$ onto all of $\mathfrak{C}'$. □

In 2011, Jones made a big leap by providing the following extension to the Guthrie-Nymann set which also yielded a continuum of subsum sets yielding Cantorvals.

Theorem 8. *(Jones) The set of subsums*

$$\sum_{i=1}^{\infty}\left[\varepsilon_{4i-3}\left(\frac{3}{q^i}\right)+\varepsilon_{4i-2}\left(\frac{2}{q^i}\right)+\varepsilon_{4i-2}\left(\frac{2}{q^i}\right)+\varepsilon_{2i}\left(\frac{2}{q^i}\right)\right],\ \varepsilon_i \in \{0,1\},\ \forall i \in \mathbb{N}$$

is a cantorval for all q such that

$$\frac{1}{5} \leq \sum_{i=1}^{\infty} < \frac{2}{9}.$$

But then in 2015, inspired by Kenyon[**?**], Nitecki[7] paved the way for the next big leap.

Theorem 9. *(Kenyon/Nitecki). Suppose we are given $n \in \mathbb{N}$ and n integers $d_0, d_1, \ldots, d_{n-1}$ such that*

$$d_i \equiv j \mod n$$

Then the set of "generalized base \$n\$ expansions" using these "digits"

$$\mathcal{S} = \left\{\sum_{i=1}^{\infty} \frac{a_i}{n^i} | a_i \in \{d_0, \ldots, d_{n-1}\}\right\}$$

has nonempty interior.

Leading Bartoszewics et al.[1] to find the following result.

Theorem 10. *(Bartoszewics)Let $k_1 \geq k_2 \geq \cdots \geq k_m$ be positive integers and $K = \sum_{i=1}^{m} k_i$ Assume that there exist positive integers n_0 and n such that each of numbers $n_0, n_0+1, \ldots, n_0+n$ can be obtained by summing up the numbers $k_1, k_2, \ldots, k_m$ (i.e. $n_0 + j = \sum_{i=1}^{m} \varepsilon_i k_i$ with $\varepsilon_i \in \{0,1\}, j = 1, \ldots, n$). If $q \geq \frac{1}{n+1}$ then $E(k_1, \ldots, k_m; q)$ has a nonempty interior. If $q < \frac{k_m}{K+k_m}$ then $E(k_1, \ldots, k_m; q)$ is not a finite union of intervals. Consequently, if*

$$\frac{1}{n+1} \leq q < \frac{k_m}{K+k_m}$$

then

$$E(k_1, \ldots, k_m; q)$$

is a Cantorval.

Also introducing a new notation in the following way: For any $q \in \left(0, \frac{1}{2}\right)$ the symbol $(k_1, k_2, \ldots, k_m; q)$ will be used to denote the sequence $\left(k_1, k_2, \ldots, k_m, k_1 q, k_2 q, \ldots, k_m q, k_1 q^2, k_2 q^2, \ldots, k_m q^2, \ldots\right.$

Part 2. More Cantor Set Theory

Definition 11. Let

$$r = \frac{|J_{0^{(n)}}|}{|J_{0^{(n-1)}}|},$$

then

$$\mathfrak{C}_r$$

will be the Cantor set with **rate of dissection** r at the nth step.

Note that in this paper we only deal with central cantor sets having constant ratio of dissection.

3. Properties

In the literature the authors all seem to justify that $\mathfrak{C}_{\frac{1}{3}}$ is perfect by saying that the property of being perfect is preserved under nested intersection. Nitecki makes this claim in three of his papers[5, 6, 7]. This however is not true, take for example

$$\cap_{i=1}^{\infty} [-n, n] = \{0\}$$

which is clearly not perfect as it is a singleton.

Theorem 12. $\mathfrak{C}_{\frac{1}{3}}$ is perfect.

Proof. Recall that

$$\mathfrak{C}_{\frac{1}{3}} = \bigcap_{n=1}^{\infty} \mathfrak{C}_n,$$

where

$$\mathfrak{C}_n = \bigcup_{\xi_1 \cdots \xi_n \in \{0,1\}^n} J_{\xi_1 \cdots \xi_n},$$

and $\mathfrak{C}_{n+1} \subset \mathfrak{C}_n$.

Since $\{0,1\}$ is compact, it follows by Tychonoff's theorem that the set $\{0,1\}^{\mathbb{N}}$ is compact, and Since $\mathfrak{C}_{\frac{1}{3}}$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$, it follows that $\mathfrak{C}_{\frac{1}{3}}$is compact. Finally by the Heine–Borel theorem $\mathfrak{C}_{\frac{1}{3}}$ is also closed.

Now, since $J_{0^{(n)}} = \left[0, R_n = \left(\frac{1}{3}\right)^n\right]$ it follows that $|J_{\xi_1 \cdots \xi_n}| = \left(\frac{1}{3}\right)^n$. Now, choose n such that $\left(\frac{1}{3}\right)^n < \epsilon$, and let $x \in \mathfrak{C}_{\frac{1}{3}} \bigcap J_{\xi_1 \cdots \xi_n}$. If we remove the middle third from $J_{\xi_1 \cdots \xi_n}$, then $x \in \mathfrak{C}_{\frac{1}{3}} \bigcap J_{\xi_1 \cdots \xi_n 0}$ or $x \in \mathfrak{C}_{\frac{1}{3}} \bigcap J_{\xi_1 \cdots \xi_n 1}$. Assume without loss of generality that $x \in \mathfrak{C}_{\frac{1}{3}} \bigcap J_{\xi_1 \cdots \xi_n 0}$ then there exists $y \in \mathfrak{C}_{\frac{1}{3}} \bigcap J_{\xi_1 \cdots \xi_n 1}$such that $y \neq x$, thus it follows that

$$\begin{aligned} (y,x) &\subseteq J_{\xi_1 \cdots \xi_n} \\ \Rightarrow |y-x| &\leq |J_{\xi_1 \cdots \xi_n}| \\ &< \epsilon \end{aligned}$$

□

4. Cantor Set Arithmetic

The following ideas are ready to be introduced, and will help tie together the cantor sets, Cantorvals, and the series that led to them.

Definition 13. For any set $S \subset \mathbb{R}$, and $n \subset \mathbb{N}$, we let

$$\oplus_n S = \{s_1 + \ldots + s_n : s_i \in S(i = 1, \ldots, n)\}$$

denote the algebraic sum of n copies of S.

Theorem 14. $\oplus_2 \mathfrak{C}_{\frac{1}{3}} = \mathfrak{C}_{\frac{1}{3}} + \mathfrak{C}_{\frac{1}{3}} = [0,2]$

Proof. Let $c \in \mathbb{I}$. Then for all c its ternary expansion is,

$$c = \sum_{i=1}^{\infty} \frac{c_i}{3^i},\, c_i \;\in\; \{0,1,2\}$$

Let $c_i = a_i + b_i$ where $a_i, b_i \in \mathbb{N}$. It follows that

$$\begin{aligned} \mathbb{I} &= \sum \left\{\frac{c_i}{3^i}\right\}_{i=1}^{\infty},\; c \;\in\; \{0,1,2\} \\ &= \sum \left\{\frac{a_i}{3^i} + \frac{b_i}{3^i}\right\}_{i=1}^{\infty},\; a_i, b_i \;\in\; \{0,1\} \\ &= \sum \left\{\frac{a_i}{3^i}\right\}_{i=1}^{\infty} + \sum \left\{\frac{b_i}{3^i}\right\}_{i=1}^{\infty} \\ &= \frac{1}{2}\mathfrak{C} + \frac{1}{2}\mathfrak{C} \end{aligned}$$

therefore

$$\begin{aligned} \mathfrak{C} + \mathfrak{C} &= 2\mathbb{I} \\ &= [0,2] \end{aligned}$$

□

Theorem 15. $\ominus_2 \mathfrak{C}_{\frac{1}{3}} = \mathfrak{C}_{\frac{1}{3}} - \mathfrak{C}_{\frac{1}{3}} = [-1,1]$

Proof. .

$$\begin{aligned}
\mathfrak{C}_{\frac{1}{3}} - \mathfrak{C}_{\frac{1}{3}} &= \left(\mathfrak{C}_{\frac{1}{3}} - \frac{1}{2}\right) + \left(\frac{1}{2} - \mathfrak{C}_{\frac{1}{3}}\right) \\
&= \left(\mathfrak{C}_{\frac{1}{3}} - \frac{1}{2}\right) + \left(\mathfrak{C}_{\frac{1}{3}} - \frac{1}{2}\right) \\
&= \mathfrak{C}_{\frac{1}{3}} + \mathfrak{C}_{\frac{1}{3}} - 1 \\
&= [0,2] - 1 \\
&= [-1,1]
\end{aligned}$$

□

5. Basic Operations

Theorem 16. *Let $\mathfrak{C}$ be a Cantor set, and $a, b \in \mathbb{R}$, then*

(1) $\mathfrak{C} + a = \mathfrak{C}$
(2) $a\mathfrak{C} + b\mathfrak{C} = a\mathfrak{C} + (-b)\mathfrak{C} + b\sup(\mathfrak{C})$
(3) $a\mathfrak{C} + b\mathfrak{C} = a\left(\mathfrak{C} + \frac{b}{a}\mathfrak{C}\right)$

6. Algebraic Sum of Subsum Sets

In 1995 Nymann[8] proposed the following interesting theorem about the algebraic sums of subsum sets $\sum \{x_i\}_{i=1}^{\infty}$.

Theorem 17. *(Nymann) There is a positive integer m for which $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is a finite union of intervals if and only if*

$$\limsup \frac{x_n}{R_n} < \infty.$$

Moreover, the smallest positive integer for which $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is a finite union of intervals is the smallest integer m such that $x_n/R_n \le m$ for all but a finite number of integers n.

Proof. Proof: Let $m \in \mathbb{N}$ be fixed. Construct $\{c_n\}$ as follows

$$c_{(q-1)m+1} = c_{(q-1)m+2} = \cdots = c_{qm} = a_q$$

for $q = 1, 2, 3, \ldots$ We observe that

$$\sum_{i=1}^{\infty} c_i$$

converges for all n, if $0 < c_{n+1} \le c_n$, and $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is also the set of subsums of

$$\sum_{i=1}^{\infty} c_i$$

Now, by Kakeya's theorem, $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is a finite union of intervals if and only if $c_n \le R_n$ for all but finitely many index values. If $m \ne nq$,for some q, them this inequality is true. If $n = mq$, then $c_n = a_q$ and $R_n = mr_q$. Thus $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is an interval if and only if $a_q \le mr_q$ for all but a finite number of integer values of q, and the conclusion follows. □

It follows from the theorem that the smallest integer m such that $x_n/R_n \leq m$ for all n is the smallest integer for which $\oplus_m x(\varepsilon)$ is an interval, and in the same paper Nymann[8] provides the following example:

Example 18. The Cantor Ternary set is the set of subsums of $\sum x_n$, where $x_n = 2/3^n$. By the previous theorem,

$$\frac{x_n}{r_n} = \frac{2/3^n}{\sum_{k=n+1}^{\infty} 2/3^k} = 2$$

Thus, once again showing$\mathfrak{C} + \mathfrak{C} = \oplus_2 \mathfrak{C}$ is an interval.

Theorem 19. *Assume $a_{n+1}/a_n < x \leq 1$ for all n. Then $E + xE$ is an interval if and only if*

$$\frac{(a_n - r_n)}{r_{n-1}} \leq x \leq \frac{r_n}{(a_n - r_n)}$$

for all n for which $a_n > r_n$.

Now, we can talk about cantor sets in terms of the the subsums of definition 1, and analogous to theorem 11, the following theorem is formulated.

Theorem 20. *Let $a, b \epsilon \mathbb{R}$, then*

1. $\sum \{x_i\}_{i=1}^{\infty} + \sum_{i=1}^{\infty} x_n = \sum \{x_i\}_{i=1}^{\infty}$
2. $a\left(\sum \{x_i\}_{i=1}^{\infty}\right) + b\left(\sum \{x_i\}_{i=1}^{\infty}\right) = a\left(\sum \{x_i\}_{i=1}^{\infty}\right) + (-b)\left(\sum \{x_i\}_{i=1}^{\infty}\right) + b\sum_{i=1}^{\infty} x_n$
3. $a\left(\sum \{x_i\}_{i=1}^{\infty}\right) + b\left(\sum \{x_i\}_{i=1}^{\infty}\right) = a\left[\left(\sum \{x_i\}_{i=1}^{\infty}\right) + \frac{b}{a}\left(\sum \{x_i\}_{i=1}^{\infty}\right)\right]$

Part 3. Bringing It Together

Theorem 21. *(Agüero) For all integers $n \geq 3$,*

$$\frac{1}{n-1}\mathfrak{C}_{\frac{1}{n}} \quad = \quad \sum_{i=1}^{\infty} \frac{\varepsilon_i}{n^i},\ \varepsilon_i \in \{0, 1\}.$$

Theorem 22. *(Agüero) For all integers $n \geq 3$,*

$$\oplus_{n-1} \mathfrak{C}_{\frac{1}{n}} = [0, n-1]$$

Proof. Let $b \in \mathbb{I}$. Then for all b its n-nary expansion is,

$$b = \sum_{i=1}^{\infty} \frac{b_i}{3^i},\ b_i \ \in \ \{0, 1, 2, \ldots, n-1\}$$

Let $b_i = a_{i_1} + a_{i_2} + \cdots + a_{i_{n-1}}$ where $a_i, b_i \in \mathbb{N}$. It follows that

$$\begin{aligned}
\mathbb{I} \quad &= \quad \sum \left\{ \frac{b_i}{n^i} \right\}_{i=1}^{\infty},\ c \ \in \ \{0, 1, 2, \ldots, n-1\} \\
&= \quad \sum \left\{ \frac{a_{i_1}}{n^i} + \cdots + \frac{a_{i_{n-1}}}{n^i} \right\}_{i=1}^{\infty},\ a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}} \ \in \ \{0, 1\} \\
&= \quad \sum \left\{ \frac{a_{i_1}}{n^i} \right\}_{i=1}^{\infty} + \cdots + \sum \left\{ \frac{a_{i_{n-1}}}{n^i} \right\}_{i=1}^{\infty} \\
&= \quad \frac{1}{n-1}\mathfrak{C}_{\frac{1}{n}} + \cdots + \frac{1}{n-1}\mathfrak{C}_{\frac{1}{n}}
\end{aligned}$$

therefore

$$\begin{aligned}\mathfrak{C}_{\frac{1}{n}} + \cdots + \mathfrak{C}_{\frac{1}{n}} &= (n-1)\mathbb{I} \\ &= [0, n-1]\end{aligned}$$

□

Theorem 23. *(Agüero) If for integers $k_1 \geq k_2 \geq \cdots \geq k_m$,*

$$\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n = \{0, 1, \ldots, n-1\},$$

then

$$\left(\sum \left\{\left(\varepsilon_{m(t-1)+1} k_1 + \varepsilon_{m(t-1)+2} k_2 + \cdots + \varepsilon_{mt} k_m\right) q^i\right\}_{i=1}^\infty\right),\ \varepsilon_i \in \{0,1\},$$

is a Cantorval if

$$\frac{1}{n+1} \leq q < \frac{k_m}{K + k_m}.$$

Proof. Let

$$p_j \equiv j \mod n,$$

and

$$\begin{aligned}\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n &= \{0, 1, \ldots, n-1\}, \\ &= \{p_0 \mod n, p_1 \mod n, \ldots, p_{n-1} \mod n\}. \\ &\quad \{n_0 + p_0 \mod n, n_0 + p_1 \mod n, \ldots, n_0 + p_{n-1} \mod n\},\end{aligned}$$

where $n_0 = 0$. Thus, by Bartoszewics' theorem,

$$\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n = \{p_0 \mod n, p_1 \mod n, \ldots, p_{n-1} \mod n\}$$

is a Cantorval. However, by Kenyon's and Nitecki's theorem so is,

$$\sum \{\alpha_i k_i\}_{i=1}^m = \{p_0, p_1, \ldots, p_{n-1}\}.$$

....

Let $k_1 \geq k_2 \geq \cdots \geq k_m > 0$, and n be integers, then by the division algorithm there exists integers q_j and r_j such that $k_j = nq_j + r_j$ and $0 \leq r_j < n$. Now,

$$\begin{aligned}\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) &= \left(\sum \{\alpha_i n q_i + \alpha_i r_i\}_{i=1}^m\right),\ \alpha_i \in \{0,1\} \\ &= \left(\sum \{\alpha_i n q_i\}_{i=1}^m + \sum \{\alpha_i r_i\}_{i=1}^m\right)\end{aligned}$$

If

$$max\left\{\sum \{\alpha_i r_i\}_{i=1}^m\right\} < n$$

then

$$\sum \{\alpha_i n q_i\}_{i=1}^m + \sum \{\alpha_i r_i\}_{i=1}^m = \sum \{\alpha_i n q_i\}_{i=1}^m + \left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n$$

but $\sum \{\alpha_i n q_i\}_{i=1}^m$ is a finite set, so it can written in the form

$$\begin{aligned}\{s_0, s_1, \cdots, s_f\} &= \{s_0, s_0 + t_1, s_0 + t_2, \cdots, s_0 + t_f\} \\ &= \{s_0\} + \{0, t_1, t_2, \cdots, t_f\}\end{aligned}$$

where $\min \sum \{\alpha_i n q_i\}_{i=1}^m = s_o$. Also,

$$\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n = \{0, 1, \ldots, n-1\},$$

therefore,

$$\begin{aligned} &= \sum \{\alpha_i n q_i + \alpha_i n q_i'\}_{i=1}^m + \left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n \\ &= \{s_0\} + \{0, t_1, t_2, \cdots, t_f\} + \{0, 1, \ldots, n-1\} \\ &= \{0, t_1, t_2, \cdots, t_f\} + \{s_0, s_0+1, \ldots, s_0+(n-1)\} \end{aligned}$$

However, if

$$max \left\{\sum \{\alpha_i r_i\}_{i=1}^m\right\} \geq n$$

the apply the division algorithm one more time yielding

$$\begin{aligned} \sum \{\alpha_i n q_i\}_{i=1}^m + \sum \{\alpha_i r_i\}_{i=1}^m &= \sum \{\alpha_i n q_i\}_{i=1}^m + \sum \{\alpha_i n q_i'\}_{i=1}^m + \sum \{\alpha_i r_i'\}_{i=1}^m \\ &= \sum \{\alpha_i n q_i + \alpha_i n q_i'\}_{i=1}^m + \sum \{\alpha_i r_i'\}_{i=1}^m \\ &= \sum \{\alpha_i n q_i + \alpha_i n q_i'\}_{i=1}^m + \left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n \end{aligned}$$

but $\sum \{\alpha_i n q_i + \alpha_i n q_i'\}_{i=1}^m$ is a finite set, so it can written in the form

$$\begin{aligned} \{s_0, s_1, \cdots, s_f\} &= \{s_0, s_0+t_1, s_0+t_2, \cdots, s_0+t_f\} \\ &= \{s_0\} + \{0, t_1, t_2, \cdots, t_f\} \end{aligned}$$

where $\min \sum \{\alpha_i n q_i + \alpha_i n q_i'\}_{i=1}^m = s_o$. Also,

$$\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n = \{0, 1, \ldots, n-1\},$$

therefore,

$$\begin{aligned} &= \sum \{\alpha_i n q_i + \alpha_i n q_i'\}_{i=1}^m + \left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n \\ &= \{s_0\} + \{0, t_1, t_2, \cdots, t_f\} + \{0, 1, \ldots, n-1\} \\ &= \{0, t_1, t_2, \cdots, t_f\} + \{s_0, s_0+1, \ldots, s_0+(n-1)\} \end{aligned}$$

Hence, by Bartoszewics' theorem, If for integers $k_1 \geq k_2 \geq \cdots \geq k_m$,

$$\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n = \{0, 1, \ldots, n-1\},$$

it follows that,

$$\left(\sum \left\{\left(\varepsilon_{m(t-1)+1} k_1 + \varepsilon_{m(t-1)+2} k_2 + \cdots + \varepsilon_{mt} k_m\right) q\right\}_{i=1}^\infty\right), \ \varepsilon_i \in \{0, 1\},$$

is a Cantorval if

$$\frac{1}{n+1} \leq q < \frac{k_m}{K + k_m}.$$

□

Theorem 24. *(Agüero) If for integers $k_1 \geq k_2 \geq \cdots \geq k_m$,*

$$\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \mod n = \{0, 1, \ldots, n-1\},$$

then

$$k_1 \mathfrak{C}_{\frac{1}{n}} + k_2 \mathfrak{C}_{\frac{1}{n}} + \cdots + k_m \mathfrak{C}_{\frac{1}{n}},$$

is a Cantorval if

$$\frac{1}{n+1} \leq q < \frac{k_m}{K+k_m}$$

Proof. Recall that, $\frac{1}{n-1}\mathfrak{C}_{\frac{1}{n}} = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{n^i},\ \varepsilon_i \in \{0,1\}$. Then

$$\begin{aligned} k_1\mathfrak{C}_{\frac{1}{n}} + k_2\mathfrak{C}_{\frac{1}{n}} + \cdots + k_m\mathfrak{C}_{\frac{1}{n}} &= \oplus_{i=1}^{m} k_i(n-1)\sum_{i=1}^{\infty}\frac{\varepsilon_i}{n^i},\ \varepsilon_i \in \{0,1\} \\ &= (n-1)\oplus_{i=1}^{m} k_i \sum_{i=1}^{\infty}\frac{\varepsilon_i}{n^i},\ \varepsilon_i \in \{0,1\} \end{aligned}$$

which, by the previous theorem, is a Cantorval if

$$\frac{1}{n+1} \leq q < \frac{k_m}{K+k_m}$$

. □

Now that we are bringing the material together to obtain new results we observe that we are able to use the Cantor set equvalent when working with series, or vice versa, hence allowing us to produce new results. Take for example the Guthrie-Nymann set

$$T = \sum_{i=1}^{\infty}\left[\varepsilon_{2i-1}\left(\frac{3}{4^i}\right) + \varepsilon_{2i}\left(\frac{2}{4^i}\right)\right],\ \varepsilon_i \in \{0,1\},\ \forall i \in \mathbb{N}$$

Using Bartoszewics' notation,

$$T = \left(3, 2; \frac{1}{4}\right).$$

We will now extend this result.

Upon closer observation we see that

$$\left(3, \ldots, 3, 2, \ldots 2; \frac{1}{3k+2k'-1}\right)$$

for $k, k' \in \mathbb{N}$ is a cantorval. First, we notice that the linear combinations of the terms are a linear of 2's and 3's. In 1884 Alexander[9] showed that if the $\gcd(a,b) = 1$, then every consecutive integer greater than $ab - a - b$ is a linear combination of a and b. For $a = 3$ and $b = 2$ we have $\gcd(3,2) = 1$ therefore every consecutive integer $n > 1$, $n = 3k + 2k'$. Now, fix k and k', then for every $m = 3q + 2q'$, $n - m$ is a linear combination of 2's and 3's with the exception of when $n = 1$, again by alexander's result. Hence,

$$\left(\sum \{3i + 2j\}\right),\ i = 1, \ldots, k \text{ and} j = 1, \ldots, k' = \{0, 2, 3, \ldots, k + 2k' - 1, k + 2k'\}.$$

It follows that,

$$\{0, 2, 3, \ldots, k + 2k' - 1, k + 2k'\} \mod (3k + 2k' - 1) = \{1, 2, \ldots, k + 2k' - 2\},$$

and by theorem 24

$$\left(3, \ldots, 3, 2, \ldots 2; \frac{1}{3k+2k'-1}\right)$$

is a Cantorval.

In summary, the result can be restated as follows.

Theorem 25. *(Agüero)[The extended, and generalized, Guthrie-Nymann set] For $k, k' \in \mathbb{N}$,*

$$\left(3, \ldots, 3, 2, \ldots 2; \frac{1}{3k+2k'-1}\right),$$

is a Cantorval.

Proposition 26. *(Agüero) For $r \in \left(\frac{1}{n}, \frac{1}{n-1}\right)$,*

$$\oplus_{n-1}\mathfrak{C}_r = [0, n-1]$$

Part 4. **Future Work**

Here are some things we know, and some we don't:

(1) All known examples of sequences generating Cantorvals as their achievement sets have been multigeometric sequences, the simplest which was found by (Guthrie-Nymann)[3], namely,

$$T \quad = \quad \sum_{i=1}^{\infty}\left[\varepsilon_{2i-1}\left(\frac{3}{4^i}\right) + \varepsilon_{2i}\left(\frac{2}{4^i}\right)\right],\ \varepsilon_i \in \{0,1\},\ \forall\, i \in \mathbb{N}$$

(2) It is not known whether all multigeometric sequences generating Cantorvals have been discovered.

(3) We do not yet know if only multigeometric sequences produce Cantorvals, or if others exist.

(4) Many conjectures remain without proof or counterexample.

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