2023 HAWAII UNIVERSITY INTERNATIONAL CONFERENCES
SCIENCE, TECHNOLOGY, ENGINEERING, ARTS, MATHEMATICS & EDUCATION JUNE 7 - 9, 2023
PRINCE WAIKIKI RESORT, HONOLULU, HAWAII

DYNAMICALLY RELATIVE VALUE AT RISK, RELATIVE EXPECTED SHORTFALL AND FREQUENCY EQUIVALENT LEVEL OF VAR AND ES WITH TWO CORRELATIVE STOCHASTIC PROCESSES

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Dynamically Relative Value at Risk, Relative Expected Shortfall and Frequency Equivalent of VaR and ES with Two Correlative Stochastic Processes

Abstract

The calculation of dynamic value at risk (VaR), when cumulative investment and interest rate are correlative stochastic processes, is discussed. The dynamically relative value at risk (RVaR) is presented. Analysis of sensitivity of relative value at risk to the change of important parameters is carried out with the help of Monte Carlo simulation. We also discuss dynamically expected shortfall (ES) and relative expected shortfall (RES). Comparison between VaR, RVaR, ES and RES is illustrated with an example. Finally, we study how to determine dynamic frequency equivalent level with an example.

Key Words: Dynamically relative value at risk; Dynamically relative expected shortfall; Monte Carlo simulation; correlative stochastic processes; Dynamic frequency equivalent level

Introduction

In the 50's years of last century, Markowitz (1952) presented modern portfolio theory. The theory uses standard deviation as the measure of risk. However, people find that standard deviation lets both positive and negative deviation as the measure of risks. It is biased. Actually, what people concern is negative deviation. Another problem is that people hope to use loss but not deviation to measure risks. As a result, the concept of value at risk is born at the right moment. Value at risk means the maximum loss in most bad condition predicted during a period of time, in normal market condition and at a certain confidence level. There are two kinds of methods to calculation value at risk. One is analysis method and another is Monte Carlo simulation (Jorion, 2001). Value at risk is one of most important risk measure. Linsmeier and Pearson (2000) explain the concept and methodology of VaR, They also discuss the advantage and disadvantage of the methods for calculating VaR. Klugman et al.(2012) show readers how to combine loss models and credibility-based pricing models, and how to analyze loss over multiple time periods.

Since Basel Accord is introduced, value at risk has become base and standard for financial institutions to carry out risk management and regulation of financial institutions. However, several studies assume that the investment return of investment portfolio follows multi-dimensional normal distribution. In fact, the distribution of the return of investment portfolio has characteristic of fat tail. That is, the data in tail is more than that of normal distribution. Using normal distribution to calculate value at risk will produce deviation. Moreover, the calculation of value at risk has another problem, that is, it does not consider interest rate as a stochastic process, therefore, it neglects the impact of interest risk on cumulative investment return and on value at risk. In addition, since value at risk is the absolute magnitude of the loss, it cannot be used to comparison between different probability distributions (stochastic processes). Mao (2008) discusses relative value at risk in single period when investment and interest rate are correlative stochastic processes. In this article, we present the calculation of dynamic value at risk and relative value at risk when cumulative investment and interest rate are two correlative stochastic processes in multi-periods. We also carry out numerical analysis with the help of Monte Carlo simulation.

Wang and Li (2021) introduce a new distributional index, the probability equivalence level of value at risk -VaR and expected shortfall- ES (PELVE), which identifies the balancing point for the equivalence. They point out that both value at risk and expected shortfall are most popular risk measure used in banking and insurance, And they are widely applied for regulatory capital calculation, decision making, performance analysis and risk management.

Expected shortfall measures the average losses over the defined threshold (typically set as the VaR at a given confidence level α). In other words, expected shortfall is a conditional mean value, given that the loss exceeds the $(1-\alpha)$ percentile. It is also often called tail value at risk (TVaR) or expected shortfall (ES). Faroni et al., (2022) and Acerbi et al. (2001) examine the properties of Expected Shortfall in financial management. This measure is indeed demonstrated to have much better properties than VaR. They indicate that unlike VaR, ES is in general sub additive and therefore it is a Coherent Measure of Risk. Acerbi and Tasche (2002) compare some of the definitions of Expected Shortfall. They point out that there is one which is robust in the sense of generating a coherent risk measure regardless of the underlying distributions. Moreover, this Expected Shortfall can be estimated effectively even in cases where the usual estimators for VaR fail. Fuchs et al. (2017) prove several results on quantile or spectral risk measures and their domain with special consideration of the expected shortfall. They also present a particularly short proof of the sub additivity of expected shortfall. Wang (2020) study the axiomatic foundation for the expected shortfall. He presents four axioms for portfolio risk assessment. They are monotonicity, law invariance, prudence and no reward for concentration. It uniquely characterizes the family of ES. Wang (2020) further points out as a unique and most important characteristic, ES rewards portfolio diversification and penalize risk concentration. In this paper, we also discuss the approximate calculation and sensitivity analysis of dynamically expected shortfall and relative expected shortfall with two correlative stochastic processes of cumulative investment return and interest rate and with the help of Monte Carlo simulation. Finally, we study how to determine frequency equivalent level of VaR and ES.

The external economic and social environment is uncertain and change with time. We use dynamic value at risk, relative value at risk, dynamically expected shortfall and relative expected shortfall to describe the regular pattern of these variables with time and forecast their future values so as to take preventive measure ahead of time. In this way, it can reduce or even avoid extreme disasters to occur. The remaining is organized as follows. The dynamic value at risk and relative value at risk with two corrective stochastic processes is defined and sensitivity analysis for VaR and RVaR is carry out in Section 2. In Section 3, dynamically expected shortfall (ES) and relative expected shortfall (RES) are discussed, and sensitivity analysis is conducted. Section 4 discusses how to determine frequency equivalent level of VaR and ES. Section 5 concludes the paper.

2. Dynamic Value at Risk and Relative Value at Risk

2.1 The calculation of dynamic value at risk with two correlative stochastic processes

Assume that cumulative investment is D_t, and the stochastic differential equation of D_t under equilibrium martingale measure can be expressed as

$$dD_t = rD_t dt + \sigma_1 D_t dw_1 , \tag{1}$$

where the interest rate follows an Ornstein-Uhlenbeck mean reverting stochastic process (Cox, Ingersoll and Ross, 1985):

$$dr_t = \kappa(\mu - r_t)dt + \sigma\sqrt{r_t}dw_2 \tag{2}$$

In equations (1) and (2) w_1 and w_2 are two correlative Wiener processes with instant correlation coefficient ρ_{12} , σ denotes the volatility of interest rate, σ_1 indicates the volatility of investment, μ is the long run equilibrium interest rate, the gap between long run equilibrium and current level is presented by μ-r and κ is a measure of the sense of urgency exhibited in financial markets to close the gap and gives the speed at which the gap is reduced, where the speed is expressed in annual terms.

Assume that the return rate of investment at time *t* is R_t, then

$$R_t = \frac{dD_t}{D_t} = rdt + \sigma_1 dw_1 \tag{3}$$

By simulation, we can find empirical distribution. And further, we can obtain the value at risk at confidence level, $1\text{-}\alpha$. That is

$$VaR_\alpha(T) = ER_T - R_{T,\alpha}, \tag{4}$$

where ER_T expresses the expected return rate of cumulative investment during period of forecasting, *T*. It is substituted by the means of samples. $R_{T,\alpha}$ is lowest return rate during period of forecasting. That is,

$$P\left\{R_T \geq R_{T,\alpha}\right\} = 1-\alpha, \tag{5}$$

and $R_{T,\alpha}$ can be obtained through simulation of empirical distribution of return rate. We can calculate the value of VaR with the help of Monte Carlo simulation if the parameters of $\kappa, \mu, r_0, \sigma, \sigma_1, \rho_{12}, T$ are given.

2.2 Calculation of dynamically relative value at risk

There is a problem when comparing value at risk between different probability distributions (stochastic processes), that is, VaR only reflects absolute level of value at risk, but not relative level of value at risk. We can find from equation (4) that the higher the expected return rate of investment, the higher the value at risk if other conditions keep unchanged. Since it does not reflect the value at risk of return rate of unit investment, it is not possible to compare value at risk with different probability distributions (stochastic processes). It is similar to absolute error and relative error. It is necessary to use relative error rather than absolute error when comparison of errors of different objectives. For example, there are two probability distributions of return rate of investment, where one is with 0.15 expected return rate of investment, another is with 0.25 expected return rate of investment. They have same value at risk such as 0.10, but the level of relative value at risk is different. Obviously, the relative risk level with 0.15 expected return rate of investment is greater than that with 0.25 expected return rate of investment.

Let relative value at risk be $RVaR(T)$ expressed as

$$RVaR_{\alpha}(T)=\frac{VaR_{\alpha}(T)}{ER_{T}}=1-\frac{R_{T,\alpha}}{ER_{T}} \quad . \tag{6}$$

Using Monte Carlo simulation, we can find $R_{T,\alpha}$ and ER_T , further, we can easily find the relative value at risk.

Examples: Assume that the parameters of two stochastic processes are

κ=0.5, $\sigma=0.12, \sigma_1=0.17, \mu=0.10, r_0=0.05, \rho$=0.2, $T=1,2,3,4$. Figure 1 and Figure 2 are intersecting surfaces of dynamic empirical cumulative probability distribution and density of probability at time T=1,2,3,4 respectively.

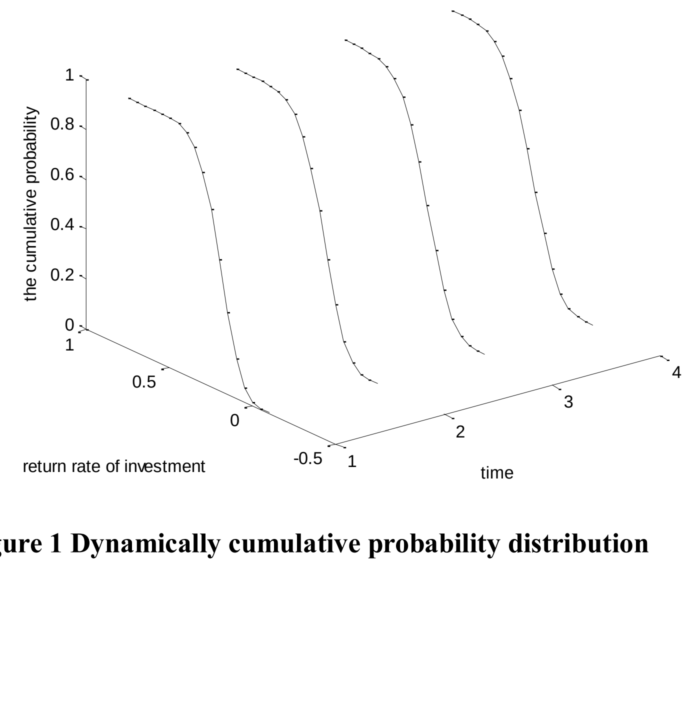

Figure 1 Dynamically cumulative probability distribution

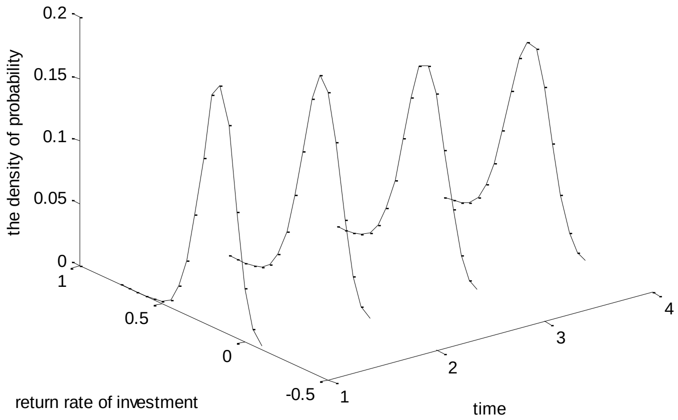

Figure 2 Dynamical density of probability

Table 1 lists the values of value at risk and relative value at risk when time term of investment takes 1,2,3,4. Figure 3 and Figure 4 describe the changes patterns of value at risk, relative value at risk and cumulative expected investment return rate. Table 1, Figure 3 and Figure 4 indicate that value at risk increases with the increase of T when $\alpha = 0.05$ and $\alpha = 0.01$ respectively. However, the relative value at risk (RVaR) decreases with the increase of T. When T changes from 1 to 4, RVaR becomes smaller and smaller. It denotes that the risk of cumulative investment return rate of per unit tends to be smaller with time. The reason is that the return rate of cumulative investment increases with the increase of T.

Table 1 Values of value at risk and relative value at risk when time term T takes different values.

κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.1997	0.2337	0.2563	0.2709
$R_{T,\ \alpha}$	0.0450	0.0610	0.0750	0.0800
$VaR_\alpha(T)$	0.1547	0.1727	0.1813	0.1900
$RVaR_\alpha(T)$	0.7747	0.7390	0.7074	0.7019
κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \alpha = 0.01, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.1997	0.2337	0.2563	0.2709

$R_{T,\ \alpha}$	-0.010	0.0080	0.0130	0.0200
$VaR_{\alpha}(T)$	0.2097	0.2257	0.2433	0.2509
$RVaR_{\alpha}(T)$	1.0500	0.9658	0.9493	0.9262

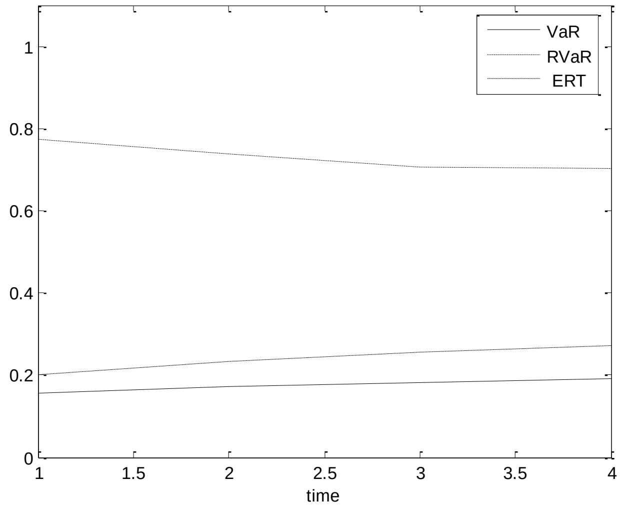

Figure 3 The change pattern of VaR and RVaR with time(α=0.05)

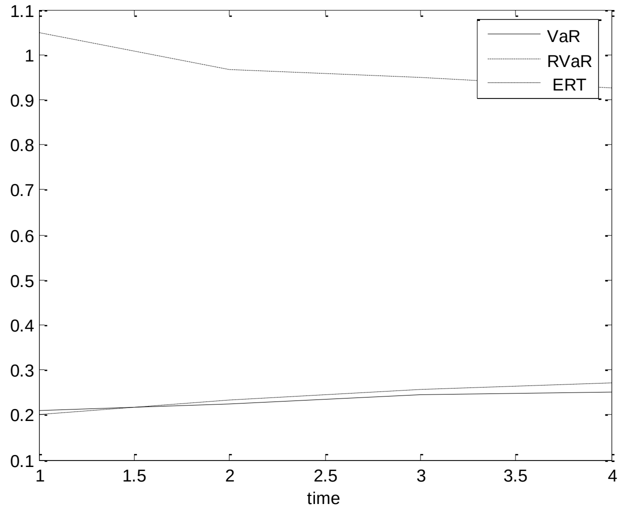

Figure 4 The change pattern of VaR and RVaR with time(α=0.01)

2.3 Sensitivity Analysis for VaR and RVaR

In the following, we will conduct sensitivity analysis when important parameters $\kappa, \sigma, \sigma_1, \mu, \rho_{12}$ increase or decrease 20%. Table 2 through Table 6 lists the results of sensitivity analysis. Table 2 through Table 6 shows that relative value at risk is most sensitive to the change of volatility of cumulative investment return and it is also rather sensitive to the change of volatility of interest rate and to the correlation coefficient between interest rate and return rate of cumulative investment. Increase of volatility of cumulative investment return will greatly increase value at risk and the relative value at risk and vise verse. And also increasing the correlation coefficient between interest rate and return rate of cumulative investment will increase both value at risk and the relative value at risk and vise verse. It is interesting to notice that increase of interest rate will decrease value at risk but increase relative value at risk. Therefore, it is important to consider the impact of the volatility of interest rate and the correlation coefficient between interest rate and volatility of return rate of cumulative investment on value at risk and relative value at risk. It is also important to decrease the risk of interest rate so as to decrease relative value at risk. For sensitive factors (parameters), it is necessary to estimate the values of these important factors (parameters) as accurate as possible. Table 2 through Table 6 also indicates that the change of parameters does not change the direction of value at risk and relative value at risk. In all cases, value at risk increases with time, however, the relative value at risk decreases with time. Regulators must consider both value at risk and relative value at risk in their regulatory decision.

Table 2 Values of value at risk and relative value at risk when $\kappa=0.5(1\pm 0.2)$

$\kappa=0.5(1+0.2), \sigma=0.12, \sigma_1=0.17, \mu=0.10, r_0=0.05, \alpha=0.05, \rho_{12}=0.2$				
T	1	2	3	4
ER_T	0.2039	0.2278	0.2530	0.2614
$R_{T,\ \alpha}$	0.0500	0.0600	0.0750	0.0800
$VaR_\alpha(T)$	0.1539	0.1678	0.1780	0.1814
$RVaR_\alpha(T)$	0.7548	0.7366	0.7036	0.6940
$\kappa=0.5(1-0.2), \sigma=0.12, \sigma_1=0.17, \mu=0.10, r_0=0.05, \alpha=0.05, \rho_{12}=0.2$				
T	1	2	3	4
ER_T	0.1940	0.2314	0.2760	0.2840
$R_{T,\ \alpha}$	0.0400	0.0580	0.0700	0.0820

$VaR_{\alpha}(T)$	0.1540	0.1734	0.2060	0.2020
$RVaR_{\alpha}(T)$	0.7938	0.7494	0.7464	0.7113

Table 3 Values of value at risk and relative value at risk when $\sigma = 0.12(1 \pm 0.2)$

$\kappa=0.5, \sigma = 0.12(1+0.2), \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.2237	0.2639	0.2937	0.3130
$R_{T,\ \alpha}$	0.0550	0.0750	0.0890	0.1000
$VaR_{\alpha}(T)$	0.1687	0.1889	0.2047	0.2130
$RVaR_{\alpha}(T)$	0.7541	0.7158	0.6970	0.6805
$\kappa=0.5, \sigma = 0.12(1-0.2), \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \rho_{12} = 0.2, \alpha = 0.05$				
T	1	2	3	4
ER_T	0.1751	0.2012	0.2253	0.2532
$R_{T,\ \alpha}$	0.0350	0.0480	0.0580	0.0680
$VaR_{\alpha}(T)$	0.1401	0.1532	0.1570	0.1748
$RVaR_{\alpha}(T)$	0.8001	0.7614	0.6968	0.6903

Table 4 Values of value at risk and relative value at risk when $\sigma_1 = 0.17(1 \pm 0.2)$

$\kappa=0.5, \sigma = 0.12, \sigma_1 = 0.17(1+0.2), \mu = 0.10, r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.2007	0.2331	0.2486	0.2915
$R_{T,\ \alpha}$	0.0250	0.0400	0.0500	0.0600
$VaR_{\alpha}(T)$	0.1757	0.1931	0.1986	0.2315
$RVaR_{\alpha}(T)$	0.8754	0.8284	0.7989	0.7942
$\kappa=0.5, \sigma = 0.12, \sigma_1 = 0.17(1-0.2), \mu = 0.10, r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.1981	0.2338	0.2541	0.2748
$R_{T,\ \alpha}$	0.0650	0.0850	0.0950	0.1050
$VaR_{\alpha}(T)$	0.1331	0.1488	0.1584	0.1698
$RVaR_{\alpha}(T)$	0.6719	0.6364	0.6234	0.6179

Table 5 Values of value at risk and relative value at risk when $\mu = 0.10(1 \pm 0.2)$

$\kappa=0.5, \sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10(1+0.2), r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.2091	0.2488	0.2682	0.2831
$R_{T,\ \alpha}$	0.0500	0.0694	0.0800	0.0900
$VaR_{\alpha}(T)$	0.1591	0.1794	0.1882	0.1931
$RVaR_{\alpha}(T)$	0.7608	0.7211	0.7017	0.6821

$\kappa=0.5, \sigma=0.12, \sigma_1=0.17, \mu=0.10(1-0.2), r_0=0.05, \alpha=0.05, \rho_{12}=0.2$				
T	1	2	3	4
ER_T	0.1891	0.2223	0.2444	0.2623
$R_{T,\ \alpha}$	0.0400	0.0550	0.0650	0.0740
$VaR_\alpha(T)$	0.1491	0.1673	0.1794	0.1883
$RVaR_\alpha(T)$	0.7885	0.7526	0.7340	0.7179

Table 6 Values of value at risk and relative value at risk when $\rho_{12}=0.2(1\pm 0.2)$

$\kappa=0.5, \sigma=0.12, \sigma_1=0.17, \mu=0.10, r_0=0.05, \alpha=0.05, \rho_{12}=0.2(1+0.2)$				
T	1	2	3	4
ER_T	0.2008	0.2326	0.2550	0.2718
$R_{T,\ \alpha}$	0.0390	0.0560	0.0660	0.0740
$VaR_\alpha(T)$	0.1618	0.1760	0.1890	0.1978
$RVaR_\alpha(T)$	0.8058	0.7567	0.7412	0.7277
$\kappa=0.5, \sigma=0.12, \sigma_1=0.17, \mu=0.10, r_0=0.05, \alpha=0.05, \rho_{12}=0.2(1\text{-}0.2)$				
T	1	2	3	4
ER_T	0.1985	0.2317	0.2534	0.2685
$R_{T,\ \alpha}$	0.0500	0.0650	0.0770	0.0870
$VaR_\alpha(T)$	0.1485	0.1667	0.1764	0.1815
$RVaR_\alpha(T)$	0.7481	0.7195	0.6961	0.6760

Since 2016, Europe union presents Insolvency II to carry out regulation of insurance companies. They use VaR as minimum capital requirement. However, VaR is positively related to the net value of assets, In fact, the insurance companies with high VaR is not necessary to have high risk. Considering both value at risk and relative value at risk can well solve this problem.

3. Dynamically Expected Shortfall and Relative Expected Shortfall

3.1 The calculation of dynamically expected shortfall and relative expected shortfall

Expected shortfall can be written as

$$\begin{aligned} ES_\alpha(T) &= E\left(R_T / R_T \geq VaR_\alpha(T)\right) \\ &= \frac{E(R_T\ \mathbf{1}R_T \geq VaR_\alpha(R_T))}{Pr\left(R_T \geq VaR_\alpha(R_T)\right)} \end{aligned} \tag{7}$$

Relative expected shortfall can be expressed as

$$RES_{\alpha}(R_T) = \frac{ES_{\alpha}(R_T)}{ER_T} \qquad (8)$$

Since VaR has no explicit expression, we can only get approximated value of ES and RES by Monte Carlo simulation.

Examples: Assume that the parameters of two stochastic processes are

κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \rho_{12}$=0.2, $T = 1, 2, 3, 4$.

Table7, Figure 5 and Figure 6 describe the change pattern of expected shortfall and relative expected shortfall with time, when α=0.01, 0.05 respectively. Table 1 and Table 7 indicate that the value of expected shortfall is greater than value at risk and relative expected shortfall is greater than relative value at risk at same time point and at same confidence level. It means that regulation is more prudent, and the financial institutions need to raise more capital in order to cope with extreme events and satisfy the regulation requirement. Please note capital is an expensive resource. Table 7 also indicates that expected shortfall increases with time, but relative expected shortfall decreases with time. It is necessary for regulator to consider relative expected shortfall besides expected shortfall in their decision.

Table 7 Values of expected shortfall and relative expected shortfall when time term T takes different values.

κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.1997	0.2337	0.2563	0.2709
$ES_{\alpha}(T)$	0.3856	0.4151	0.4266	0.4345
$RES_{\alpha}(T)$	1.9309	1.7762	1.6645	1.6039
κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \alpha = 0.01, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.1997	0.2337	0.2563	0.2709
$ES_{\alpha}(T)$	0.6478	0.6252	0.6319	0.6369
$RES_{\alpha}(T)$	3.2252	2.6290	2.3612	2.2764

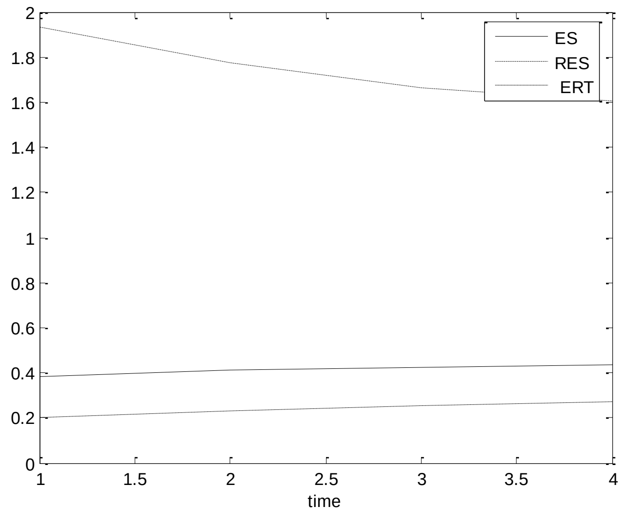

Figure 5 The change pattern of ES and RES with time (α=0.05)

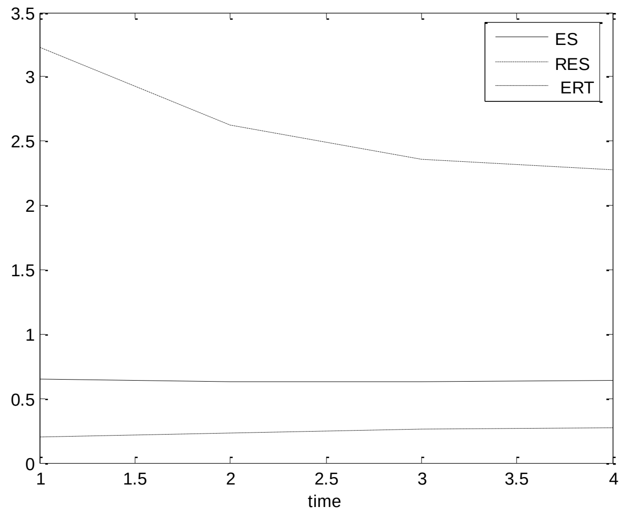

Figure 6 The change pattern of ES and RES with time (α=0.01)

3.2 Sensitive analysis for dynamically expected shortfall and relative expected shortfall

Table 8 through Table 12 lists the results of sensitivity analysis. We find from Table 8 through Table 12 that relative expected shortfall is most sensitive to the change of volatility of cumulative investment return, It is rather sensitive to the change of volatility of interest rate, return rate of long term, the speed of interest rate return back to the equivalent return rate of long term and correlation coefficient between cumulative investment return and interest rate, Increasing volatilities of interest rate will decrease both expected shortfall and relative expected shortfall. The main reason may be that the increase of volatility of interest rate will lead the increase of cumulative investment return increase. However, increasing volatilities of cumulative investment return will increase both expected shortfall and relative expected shortfall. Increase of the correlation coefficient between cumulative investment return and interest rate will increase both expected shortfall and relative expected shortfall. Therefore, it is important to consider the impact of interest rate and its correlation on cumulative investment return. We also find that increase of equivalent return rate of long term will decrease both expected shortfall and relative expected shortfall. Decreasing the speed of interest rate returning back to the equivalent return rate of long term will increase both expected shortfall and relative expected shortfall. For all cases, the relative expected shortfall decreases with time, but the expected shortfall increases with time.

Table 8 Values of expected shortfall and relative expected shortfall when κ=0.5(1$\pm$0.2)

$\kappa=0.5(1+0.2), \sigma=0.12, \sigma_1=0.17, \mu=0.10, r_0=0.05, \alpha=0.05, \rho_{12}=0.2$				
T	1	2	3	4
ER_T	0.2038	0.2282	0.2467	0.2661
$ES_\alpha(T)$	0.3820	0.4143	0.4232	0.4359
$RES_\alpha(T)$	1.8742	1.8155	1.7155	1.6380
κ=0.5(1-0.2), $\sigma=0.12, \sigma_1=0.17, \mu=0.10, r_0=0.05, \alpha=0.05, \rho_{12}=0.2$				
T	1	2	3	4
ER_T	0.1939	0.2413	0.2602	0.3078
$ES_\alpha(T)$	0.3930	0.4222	0.4290	0.4478
$RES_\alpha(T)$	2.0268	1.7490	1.6490	1.4550

Table 9 Values of expected shortfall and relative expected shortfall when .σ=0.12(1$\pm$0.2)

$\kappa=0.5, \sigma=0.12(1+0.2), \sigma_1=0.17, \mu=0.10, r_0=0.05, \alpha=0.05, \rho_{12}=0.2$				
T	1	2	3	4
ER_T	0.2288	0.2510	0.2828	0.2955
$ES_\alpha(T)$	0.3723	0.4057	0.4182	0.4323
$RES_\alpha(T)$	1.6272	1.6164	1.4787	1.4630
$\kappa=0.5, \sigma=0.12(1-0.2), \sigma_1=0.17, \mu=0.10, r_0=0.05, \alpha=0.05, \rho=0.2$				

T	1	2	3	4
ER_T	0.1795	0.2043	0.2181	0.2246
$ES_\alpha(T)$	0.4025	0.4377	0.4491	0.4694
$RES_\alpha(T)$	2.2419	2.1424	2.0585	2.0898

Table 10 Values of expected shortfall and relative expected shortfall when σ_1=0.17(1±0.2)

κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17(1+0.2), \mu = 0.10, r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.2057	0.2308	0.2497	0.2627
$ES_\alpha(T)$	0.4120	0.4472	0.4579	0.4725
$RES_\alpha(T)$	2.0032	1.9374	1.8342	1.7986
κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17(1-0.2), \mu = 0.10, r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.2031	0.2328	0.2535	0.2708
$ES_\alpha(T)$	0.3483	0.3779	0.3869	0.3974
$RES_\alpha(T)$	1.7149	1.6235	1.5264	1.4676

Table 11 Values of expected shortfall and relative expected shortfall when μ=0.10(1±0.2)

κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10(1+0.2), r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.2161	0.2380	0.2666	0.2785
$ES_\alpha(T)$	0.3700	0.4038	0.4143	0.4261
$RES_\alpha(T)$	1.7121	1.6966	1.5540	1.5375
κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10(1-0.2), r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2$				
T	1	2	3	4
ER_T	0.1923	0.2226	0.2311	0.2408
$ES_\alpha(T)$	0.3940	0.4265	0.4417	0.4577
$RES_\alpha(T)$	2.0494	1.9161	1.9113	1.9012

Table 12 Values of expected shortfall and relative expected shortfall when ρ_{12}=0.2(1±0.2)

κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2(1+0.2)$				
T	1	2	3	4
ER_T	0.1992	0.2414	0.2611	0.2716
$ES_\alpha(T)$	0.3962	0.4260	0.4332	0.4438
$RES_\alpha(T)$	1.9885	1.7643	1.6589	1.6343
κ=0.5, $\sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.2(1-0.2)$				

T	1	2	3	4
ER_T	0.1979	0.2370	0.2575	0.2820
$ES_\alpha(T)$	0.3758	0.4042	0.4136	0.4254
$RES_\alpha(T)$	1.8984	1.7054	1.6064	1.5986

4. Determination of Dynamic Frequency Equivalent level of VaR and ES

By referring to Li and Wang (2021) and Faroni et al. (2022), We present frequency equivalent level of VaR and ES which satisfies the following equivalent equation:

$$ES_{c(T)}(T) = VaR_\alpha(T) \tag{9}$$

By applying Monte Carlo simulation, we can find the quantile level of $c(T)$ which satisfies equivalent equation (9). Examples: Assume that the parameters of two stochastic processes are

$\kappa=0.5, \sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \rho_{12}=0.2, T = 1,2,3,4$

Table 13 lists the values of $c(T)$ when $\alpha=0.01$ and $0.05, T = 1,2,3,4$ respectively. Table 13 denotes that the parameter deciding frequency equivalent levels, $c(T)$ are greater than α whatever $\alpha=0.01$ or 0.05. The values of $c(T)$ decrease with time *T*. It is important to notice the errors between $VaR_\alpha(T)$ and $ES_c(T)$ is very small, the relative accuracy is greater or equal to 99.95%.

$\kappa=0.5, \sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \alpha = 0.05, \rho_{12} = 0.$				
T	1	2	3	4
$c(T)$	0.0763	0.0658	0.0611	0.0561
$VaR_\alpha(T)$	0.1547	0.1727	0.1813	0.1900
$ES_c(T)$	0.1548	0.1728	0.1815	0.1903
$RVaR_\alpha(RES_c)$	0.7765	0.7306	0.7156	0.6986
$\kappa=0.5, \sigma = 0.12, \sigma_1 = 0.17, \mu = 0.10, r_0 = 0.05, \alpha = 0.01, \rho_{12} = 0.$				
T	1	2	3	4
$c(T)$	0.0780	0.0740	0.0670	0.0655
$VaR_\alpha(T)$	0.2097	0.2257	0.2433	0.2509
$ES_c(T)$	0.2096	0.2256	0.2432	0.2510
$RVaR_\alpha(RES_c)$	1.0524	1.0041	0.9679	0.9246

Conclusions

In this paper, we present dynamically relative value at risk (value at risk per unit investment return) in multi-periods. We give the formula of dynamically relative value at risk. We carry out sensitive analysis. The results indicate that both value at risk and relative value at risk are most sensitive to the change of the volatility of cumulative investment return. And also it denotes that both value at risk and relative value at risk are rather sensitive to the volatility of interest rate and to the correlation coefficient between interest rate and return rate of cumulative investment. Increasing volatility of interest rate will decrease value at risk but increase relative value at risk. Increase of correlation coefficient between volatilities of interest rate and volatility of return rate of cumulative investment will increase both value at risk and relative value at risk. And also increasing the volatility of return rate of cumulative investment will increase both value at risk and relative value at risk. We also discuss expected shortfall and relative expected shortfall with correlative stochastic processes of interest rate and cumulative investment return. The results show that the value of expected shortfall and relative expected shortfall are greater than those of value at risk and relative value at risk at same confidence levels. It means that using expected shortfall as minimum capital requirement will require financial institutions to raise more capital at expense of more capital cost. Therefore, the criterion of regulation is a double-edged sword. It is necessary for regulators to comprehensive consider the possible loss of extreme events and high capital cost to seek a most favorable balancing point. Finally, we approach the frequency equivalent level of VaR and ES. The results show that the frequency equivalent level $c(T)$ decreases with time and it is greater than both $\alpha = 0.01$ and 0.05. Our determination of frequency equivalent level can well help regulators to balance risks and benefits of financial institutions.

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Appendix

1. Calculation of $R_{T,\alpha}, T = 1,2,3,4$

Produce two correlative stochastic processes' random number ε_1 and ε_2, which satisfies

$$\varepsilon_3 = \rho\varepsilon_1 + \sqrt{1-\rho\varepsilon_2} \ ,$$

where ε_1 and ε_2 follow standard normal distribution and ρ is correlative coefficient between two stochastic processes,

Use equations (2) and (3) and combine equation (A1) to calculate the return rate of cumulative investment $R_T, T=1,2,3,4$, where $R_T = r(T) + \sigma\sqrt{r(T)}\varepsilon_1$, (A2)

$$r(T) = r(T\text{-}1) + \kappa(\mu - r(T\text{-}1)) + \sigma_1\varepsilon_3. \quad \text{(A3)}$$

If the frequency of $R_{T,} T=1,2,3,4$ equal to α, then

we have $R_T = R_{T,\alpha}$

2. Calculation of *ES*

   Calculate $R_T, T=1,2,3,4$ by simulation and count the number satisfying $R_T \geq VaR_\alpha(T)$.

   Let this number as q and let cumulative sum of R_T be R_{1T}. By using equation (7) and calculating $ES = \dfrac{R_{1T}/q}{q/S/T}$, (A4)

   where *S* is the number of simulation, we obtain approximate value of *ES*.

3. Calculation of $c(T)$

   Let $\left|R_T - VaR_\alpha(T)\right| \leq \varepsilon$ (A5)

   For time *T*, count number $q(T)$ which satisfies in-quality (A5).

   Let $c(T)=\dfrac{q(T)}{S}$ (A6)

   Then we get approximate value of $c(T)$.