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UNDERSTANDING AND TEACHING FUNCTIONS USING A RIGOROUS APPROACH

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ABSTRACT. *Calculus is the first important course for all students who want to pursue STEM (Science, Technology, Engineering, and Mathematics) education and career. However, many high school graduates who are accepted by colleges do not understand the function concept well and are not ready to take calculus. This paper will discuss the connection between learning and teaching functions and propose a rigorous approach to teach functions for mathematics majors including pre-service/in-service teachers.*

Key Words: Functions, Inverse Functions, Mappings.

1. Preparing Teachers for the High School Classroom

The preparation of high school mathematics teachers is usually comprised of three main components which include pedagogical knowledge, mathematics content knowledge, and pedagogical content knowledge. A foundation in pedagogical knowledge is usually a part of every teacher preparation program coursework in educational foundations and education psychology, including a knowledge of adolescents, classroom management education, use of appropriate technology for the classroom, an understanding of students' social and emotional needs (SEL), and an introduction to students with special needs.

For the second component, mathematics content knowledge, the National Council of Teachers of Mathematics [15] requires recognized teacher education programs to prepare teacher candidates who can demonstrate and apply a strong background in the specific areas of number, algebra and function, calculus, discrete mathematics, linear algebra, geometry, trigonometry, measurement, and probability and statistics. Most degree programs in mathematics teacher education also require additional mathematics coursework in subjects such as basic real analysis, complex numbers, abstract algebra, and mathematical modeling. In addition to a strong knowledge base in mathematics content, pre-service teacher education programs should also focus on helping their students in the development of conceptual understanding, procedural fluency, and mathematical reasoning skills.

But what does it mean to have a foundation in these fundamental areas of mathematics? Some key skills needed by every mathematics teacher include the ability to prepare and present a lesson that helps students develop a conceptual understanding of the mathematics. According to the National Assessment of Education Progress (NAEP), "*Students demonstrate conceptual understanding in mathematics ... when they recognize, label, and generate examples of concepts; use and interrelate models, diagrams, manipulatives, and varied representations of concepts; use* *and apply principles; know and apply facts and definitions; compare, contrast, and integrate related concepts and principles; recognize interpret and apply the signs, symbols, and terms used to represent concepts"* [12]. Teachers who can help their students develop conceptual understanding, reasoning, and number sense must possess this deep understanding themselves. Mathematics teachers must also (1) have the knowledge and ability to teach classes where their students are focused on mathematics that leads to developing this conceptual understanding for themselves, and (2) support the student development of procedural fluency, or "the meaningful and flexible use of procedures to solve problems" [16]. While many educators tend to value the teaching of conceptual knowledge before students obtain procedural fluency, some researchers [1] believe that the two skills can be learned simultaneously.

Mathematics teachers must have the knowledge and ability to prepare lessons that include engaging activities focused on specific learning goals, while facilitating discussions that challenge all students. This third component, known as pedagogical content knowledge, is the knowledge of how to prepare daily lessons, design engaging activities, establish curriculum goals, lead challenging discussions, and develop assessments that provide key information about student understanding from both a formative and a summative perspective. Pre-service teachers should also be able to help students connect mathematics topics within the different subject areas of mathematics, like between algebra and statistics, and also connect mathematics to topics outside of mathematics, like to problems in physics and finance, for example.

Another essential skill needed by high school mathematics teachers is the ability to prepare standards-based lessons. This means that both the instruction and the assessments are aligned with standards written specifically for mathematics. The Common Core Standards in Mathematics (CCSS-M) and the eight mathematical practice standards known as the Standards for Mathematical Practice (SMPs) [14, 15] are currently the standards that are used in high school mathematics classrooms nationwide. An important benefit with adoption of the Common and Core is to provide a consistency and timeline of topics. The standards were written so that all necessary topics are developed during a specific grade there by placing prerequisite knowledge in the correct sequence. This timeline helps teachers understand when and to what depth a mathematical topic should be part of their lesson plans. The use of the CCSS has the potential to improve mathematics instruction nationwide and to help teachers develop rigorous and coherent instruction that help students develop deep understanding and mathematical thinking.

The topic of functions is an integral part of the mathematics curriculum and its study will begin as early as kindergarten where students are asked to study patterns of numbers and objects. Their study continues into elementary grades where they are asked to examine sequences of numbers and possibly determine the next number or object in the sequence. In middle school, they work on functions and learn ratio and proportion, linear equations, and graphs. In high school, students begin a formal study of functions that provides a foundation for more advanced mathematics such as calculus and statistics. This includes exploring a wide variety of functions that may include polynomial, exponential, logarithmic, trigonometric, quadratic, and inverse functions. A closer look at some of the CCSS that comprise functions will help the reader see the depth of knowledge needed by the teacher as they prepare instruction that develops deep understanding in their students.

Standards specific to interpreting functions expect students to understand the concept of a function and use function notations. If a teacher is clear in their own understanding of the definition of a function and how it translates into instruction that helps students develop their own concept of a function, they can provide their students with a broad knowledge for advanced study.

A set of standards in the function family asks students to analyze functions using different representations. Standard F-IF.8 states, *"Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function"* [16]. The classroom teacher should be prepared to interpret students' responses and assess if they are truly equivalent. Classroom discussion can provide students with opportunities to analyze and assess the work of their peers to see if they are in agreement with the responses. The teacher that has a strong foundation and understanding of functions will have an opportunity to lead students to use critical thinking and reasoning while providing opportunities to communicate their mathematical ideas.

In this paper, we will focus on teaching and learning the function concepts since fall 2014 in the abstract algebra class for mathematics, mathematics education, pre-service and in-service teachers including undergraduate and graduate students. There were 10-20 students in each class. Our observations and ideas are also based on our teaching and supervising experience since 2014 in the applied calculus class for undergraduates with majors in business and STEM, and the average class size is about 15 students in each class.

2. What is a function?

One of the universal concepts which appears in almost every branch of mathematics, science and technology is the concept of a function (or mapping) from one set to another set. Mastering this key concept and being able to perform operations and calculations of functions are essential for STEM majors to be successful as they take fundamental mathematics courses like calculus. Data from the Education Department's National Center for Education Statistics shows that between 2011-2012 to 2014, 52% of students whose original declared major were mathematics switched majors within 3 years and 40% of students whose originally declared major was natural science switched majors within 3 years [13]. It is a complicated phenomenon and a simple answer cannot explain why this happened. One factor is that college level mathematics is so different from high school mathematics and freshmen out of high school have no idea what mathematics in college is until they finish calculus and start taking other mathematics classes. Many freshmen were not trained to dig deep into the concepts, think logically, conduct tedious computations, and write a solution or proof rigorously by reasoning and correct/abstract mathematical symbols. In this section, we will examine the ways that the students understand the definition of a function in our classes.

Many pre-service mathematics teachers are required to take specific college mathematics courses, such as abstract algebra, before they can teach high school mathematics. However, many teachers do not believe that they can use the complex ideas learned in college to teach their students. On the surface, the content from courses like abstract algebra may seem unrelated, but recognizing important links and altering instruction could make mathematical ideas in abstract algebra and other advanced courses seem accessible to the high school students. We will not only study the connections in content, but also examine different ways to teach the content and how this can be beneficial to students. In the following sections, we will focus on one of the key concepts: functions.

For mathematics majors, the concept of a function, or mapping, is taught as a preliminary topic in most abstract algebra courses and books [8]. In K-12 mathematics education in the United States, students are introduced to functions in 8th grade and continue to build on their knowledge of functions throughout high school. Although students learn continuously about functions for years, the idea of functions as a mapping from a set to another set may seem foreign to many students. Many high school students may only be exposed to functions that are mappings from the real numbers to itself. A precise definition of a function (or mapping) is: a function from a nonempty set A to another nonempty set B is a subset C in $A \times B$ such that for every element a in A, there is a unique element b in B such that the ordered pair (a, b) is in C. This definition is not commonly used. The following is the definition of a function in the widely used calculus book of Stewart [20].

Definition 2.1. (**Stewart**) A function f is a rule that assign each element x in a set A exactly one element called $f(x)$, in a set B. The set A is called the domain of f, B is called the codomain of f and the set $\{f(x), x \in A\}$ is called the range of f.

In general, the range is a subset of B but not the entire codomain B. If we compare the definition in Stewart's book with the definition in the book *"Abstract Algebra"* by Herstein [8], they are the same.

Definition 2.2. (**Herstein**) A function or mapping f from a set S to a set T is a rule that assigns to each element s in S a unique element t in T, written $f : S \to T$, $f(s) = t$.

Two functions f and g from a set S to a set T are equal if for all x in S, $f(x) = g(x)$. The difference is that in calculus, we only consider the real valued functions so that the sets A (domain) and B are subsets of $\mathbb{R}$, the set of real numbers, whereas in abstract algebra and other advanced mathematics classes, the sets S and T are more general and can be any sets.

There are several different ways to describe a function in high school textbooks. When students are introduced to functions, they use a mapping diagram to decide if a relation is a function. This initial introduction to functions, unfortunately, is the only time a high school student relates a function to a mapping. A mapping diagram allows students to visualize a function as a relation of elements in a set, rather than only a relation between and on a Cartesian plane. Having students learn more about functions as mappings could be added into the high school curriculum to give students the ability to explore various types of functions. To determine if a graphed relation represents a function on the $xy-$plane, high school students are taught to use the Vertical Line Test: If we can draw a vertical line that intersects the graph at more than one point, then there is a specific value which is assigned to more than one value, which would make the relation not a function.

Most students in our applied calculus classes feel that they understand a function if it is expressed by the following methods: (a) a description in words; (b) a table of values; (c) a graph; and (d) an explicit formula. To test if they really know what a function is, each semester in our class, we always ask them the same question:

Let x and y be two real numbers and $x^2 + y^2 = 1$.

(1) What is the graph of this equation?

(2) Does the equation define y as a function of x?

Students are able to understand that the graph of the equation is a unit circle with center $(0, 0)$. But almost all students are confused with the second question. Then we add more questions:

(3) Does the circle pass the vertical line test?

(4) Solve y in terms of x from the equation.

Now most students can correctly answer (3) and (4) after they see that for one x value in the open interval $(-1, 1)$, there are two values of y corresponding to this x. So they know that the equation does not define y as a function of x. We then change the question to:

(5) If x and y are two real numbers such that $x^2 + y^2 = 1$ and $y \geq 0$ (the upper half circle), does the equation define y as a function of x?

After students discuss the first four questions, most students can apply the vertical line test and receive the correct answer for question (5). These questions show that more than half of students in our applied calculus classes have trouble to understand the implicit function abstractly but they feel that they know it by applying the vertical line test. Without a graph and the vertical line test, when they learn the derivative of an implicit function defined by an equation, it is very difficult for them to understand why the equation of x and y gives a function $y = f(x)$ and do not know how to compute the derivative of an implicit function $f(x)$. So the concept of an implicit function should be introduced in high school and we believe that it is approachable by many high school students in the situation that the vertical line test can be used or they can express y in terms of x.

3. Notations and Building Functions

Every semester, in the applied calculus class, after students are exposed to several examples of maximum and minimum problems, they are given the following problem: Find two numbers so that their sum is 100 and their product is maximum. Almost everyone is able to start with defining these two numbers as x and y and write an equation $x+y=100$. Most students can guess the answer without any explanation: $x=50$ and $y=50$. Less than 30% of students are able to figure out the key step: write the product function

$$f(x)=x(100-x)=-x^2+100x.$$

Once they obtain the product function, they are able to compute its derivative, find its critical point and apply the first or second derivative rule to get values of x and y. Even if students did not know calculus, with the product function $f(x)$, they should be able to identify it as a quadratic function:

$$f(x)=x(100-x)=-x^2+100x=-(x-50)^2+2500.$$

Then only when $x=50$, the function attains its maximum 2500 because the square term is a non-positive real number. In college algebra and applied calculus classes, most students have no idea to complete the square.

Students' response to the question shows that they cannot use the equation $x+y=100$ to express the product as a function of a single variable x. They were trained in high school to guess the answer, not approach the problem by reasoning and building the function from the equation.

Common and Core Standards require students to be able to *"Reason abstractly and quantitatively"*. In our applied calculus classes, students are unable to identify the similarity between the following geometric problem and the above number problem and think that they are completely different problems: Find the dimensions of a rectangle with the perimeter 200 feet that has the maximum area. But more students are able to set up the area function as a single variable function and find out that when its dimensions are 50 feet and 50 feet, the area is maximized.

It is clear that very few students' content knowledge and skills in our applied calculus class meet the following Common and Core State Standards:

1. CCSS.Math.Content.HSF.BF.A.1: *Write a function that describes a relationship between two quantities.*
2. CCSS.Math.Content.HSF.IF.B.4: *For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*
3. CCSS.Math.Content.HSF.IF.C.7.a: *Graph linear and quadratic functions and show intercepts, maxima, and minima.*
4. CCSS.Math.Content.HSF.IF.C.7.a: *Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.*

5. CCSS.Math.Content.HSF.IF.C.8.a: *Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.*

The second problem many college students have is using the notation of a function. For example, given $y = f(x) = x^2$, more than half of students in our applied calculus classes are able to calculate $f(x+1)$ but confused with $f(x+h)$, where h is a constant. Therefore they do not know how to compute $[f(x+h) - f(x)]/h$ in the definition of the limit. Some students also do not know the difference among the notations: f, $f(x)$, and $f(a)$, where a is a constant. For example, to evaluate $f(3)$, it is not uncommon that students may write

$$f(x) = x^2 = 3^2 = 9$$

and do not know the correct notations:

$$f(x)|_{x=3} = f(3) = 3^2 = 9,$$

which means that the value of the function at 3 is 9.

The third problem is that most students only pay attention to the formula of a function and are not aware that a function consists of three parts: the rule f, two sets S (domain) and T (codomain); and if any part is changed, then we have a different function. For example, everyone knows that $y = \sqrt{x}$ is a function from the set $S = \mathbb{R}^+ \cup \{0\}$ of non-negative real numbers to itself. But if we ask students if it is a function from the set of real numbers $\mathbb{R}$ to itself, then many students including mathematics majors are confused because conceptually, they are not aware that the square root of a negative number is not a real number even though they know that $\sqrt{-2}$ is not a real number.

4. One-to-one, and onto functions

By the Common and Core State Standards CCSS.Math.Content.HSF.IF.B.4, the concept of the increasing (decreasing) functions should be introduced to students in high school. More generally, a function or mapping f from a set S to a set T is injective, or one-to-one, if for two elements x_1 and x_2 in S, $f(x_1) = f(x_2)$ implies $x_1 = x_2$ [8]. In other words, f maps two distinct elements in the domain S to two distinct elements in the codomain T. Particularly, if f is a real valued function defined over the set of real numbers, we can use Horizontal Test to verify if f is one-to-one: If we can draw a horizontal line that intersects the graph at more than one point, then there are at least two different values x are assigned to the same value y, which would make the function not one-to-one or injective.

The range and codomains are two difficult concepts and the range of a function may not be equal to its codomain. The mapping or function f is onto or surjective if for every element t in T, there is an element s in S such that $f(s) = t$ [8]. So, f is onto if every element of T is the image of some elements of S under the mapping f. Or if the range of f is the entire set T, then f is onto. If a function is both one-to-one and onto, then it is called a bijective function or a one-to-one correspondence.

In the abstract algebra class, many students are able to understand a one-to one function but struggle to show that a function is onto or not onto. A typical problem they can solve is: does the function f from the set S of real numbers to itself given by $f(x) = x^3$ one-to-one and onto? Most students can write a complete proof. But if the problem is changed to: if f is the function from the set $\mathbb{Z}$ of integers to itself given by $f(x) = x^3$, then is it one-to-one and onto? More than half of students can figure out one-to-one proof but make mistakes to claim that it is still onto. They only pay attention to the function formula, always apply their knowledge of real valued functions over the set of real numbers to the functions with different domains and ranges. They did not learn in high school that in the definition of the function, the rule f, the domain S and codomain T, together with the formula (rule) of f, are three important parts of the function. If the domain set S or the codomain T is changed, then with the same function formula, the new function is different from the original one. Most students have no abstract idea that any function is an onto function from its domain to its range (e.g. $y = f(x) = 2x$ is an onto function from $\mathbb{Z}$ to $2\mathbb{Z}$) and every one-to-one function is a bijective function from its domain to its range (e.g., $y = f(x) = 3x$ is a bijective function from $\mathbb{Z}$ to $3\mathbb{Z}$) even though they can obtain the conclusion for these specific examples. It is worthwhile to emphasize these facts in class and apply them to calculus. Particularly, a strictly increasing function or a strictly decreasing function is an one-to-one function. Intuitively, students are able to identify that the following functions over the set of real numbers are strictly increasing and by the horizontal test, each function is one-to-one: (1) $y = f(x) = ax, a > 0$, (2) $y = f(x) = x^3$, (3) $y = f(x) = e^x$, (4) $y = f(x) = a^x, \quad a > 1$, (5) $y = f(x) = \ln x \quad (x > 0)$, (6) $y = f(x) = \log_a x$, $a > 1$, $(x > 0)$.

5. COMPOSITION OF FUNCTIONS

The composition of functions is a topic covered in high school. Most pre-service students in abstract algebra class are able to compute the composition, understand the proofs that (1) the composition of two one-to-one functions is still one-to-one; (2) the composition of two onto functions is still onto; therefore (3) the composition of two bijections is still a bijection. It is clear to them that the composition of two functions is not commutative. For examples, if $f(x) = x^2 + 1$, and $g(x) = x^3$, then $f(g(x)) = x^6 + 1$ and $g(f(x)) = (x^2 + 1)^3$, so $f(g(x)) \neq g(f(x))$. In general, it is not clear to most students that the composition function $f(g(x))$ is well-defined only if the range of g is contained in the domain of f. To help them understand it, let p be the projection function from the real xy-plane to x-axis, i.e., $p(x, y) = x$. Let $q(x) = x^2$. Then $q(p(x, y)) = q(x) = x^2$ makes sense but $p(q(x))$ is not defined because the range of q is set of nonnegative real numbers which are not contained in the domain of p. It is useful to draw a composition mapping diagram to show the composition:

$$S \xrightarrow{f} T \xrightarrow{g} U, \quad g \circ f : S \longrightarrow U$$

where f is a function from S to T, g is a function from T to U and $g \circ f$ is the compositin function from S to U.

We give students the following question in every abstract algebra class [8]:

Question: Let f be a function from the set $\mathbb{Z}$ of integers to itself and $f(x) = ax + b$. Find the numbers a and b such that the composition of f with itself is the identity mapping (function). Here an identity function (mapping) Id_S from a set S to itself is defined to be

$$Id_S : S \to S, \quad Id_S(x) = x$$

for every $x \in S$.

Most students are able to obtain the equation $f(f(x)) = a^2x + ab + b = x$ with minor notation mistakes and many students can guess $a = 1$ and $b = 0$. They do not know how to find all values of a and b from this equation. It implies that they do not know that two linear functions defined over $\mathbb{Z}$ are equal if they have the same constant term and same coefficient of x. They also do not know to change the equation to $(a^2 - 1)x + ab + b = 0$, then conclude that the left-hand side linear function is the zero function so $a^2 - 1 = 0$ and $ab + b = 0$. So there are two solutions: if $a = 1$, then $b = 0$ and if $a = -1$, then b can be any integer.

In calculus, it is important to recognize the structure of a composition function in order to compute its derivative by chain rule and understand the inverse function. For pre-service and in-service teachers, to understand one-to-one, onto and composition functions is the key step to understand the existence of inverse functions.

6. Inverse functions

The existance of an inverse function for a given function relies on one-to-one and onto functions. Intuitively, students can understand that a one-to-one function is a function that does not map two different elements in the domain of f to one element in its range. It is equavelent to saying that their images under this mapping f are two different elements in the range. Geometrically, they understand the horizontal test if the domain and range are subsets of real numbers. Even though most students are able to verify a given function is one-to-one by definition, but they have difficulty to check if a function is onto, partially because it is hard to find the range of a given function. Before the abstract definition and theorems of inverse functions are presented to students, it is helpful to start with the functions they learn in high school [7]. Among these functions with inverses, linear functions and $y = x^3$ over the set of real numbers are easiest examples. To verify by composition that one function is the inverse of another (CCSS.Math.Content.HSF.BF.B.4.b), we show students the examples they are able to understand.

Example 6.1. Let f be a function defined over the finite set $S = \{1, 2, 3\}$ such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$, then the inverse function f^{-1} exists and $f^{-1}(1) = 3$, $f^{-1}(2) = 1$ and $f^{-1}(3) = 2$. It is easy to compute

$$f^{-1}(f(1)) = f^{-1}(2) = 1, \quad f^{-1}(f(2)) = f^{-1}(3) = 2, \quad f^{-1}(f(3)) = f^{-1}(1) = 3.$$

So $f^{-1}(f(i)) = i = Id_S(i)$, $i = 1, 2, 3$. Similarly, $f(f^{-1}(i)) = i = Id_S(i)$, $i = 1, 2, 3$.

Example 6.2. If $y = f(x) = 2x + 1$, x is any real number, then $x = \frac{y}{2} - \frac{1}{2}$, so the inverse function $y = f^{-1}(x) = \frac{x}{2} - \frac{1}{2}$. The composition functions

$$f(f^{-1}(x)) = f(\frac{x}{2} - \frac{1}{2}) = 2(\frac{x}{2} - \frac{1}{2}) + 1 = (x-1) + 1 = x = Id_{\mathbb{R}}(x),$$

and

$$f^{-1}(f(x)) = f^{-1}(2x+1) = \frac{2x+1}{2} - \frac{1}{2} = (x + \frac{1}{2}) - \frac{1}{2} = x = Id_{\mathbb{R}}(x),$$

where $\mathbb{R}$ is the set of real numbers.

Example 6.3. If $y = f(x) = x^3$, x is any real number, then $x = \sqrt[3]{y}$, so the inverse function is $y = f^{-1}(x) = \sqrt[3]{x}$. Their compositions

$$f(f^{-1}(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x = Id_{\mathbb{R}}(x),$$

and

$$f^{-1}(f(x)) = f^{-1}(x^3) = (x^3)^{1/3} = x = Id_{\mathbb{R}}(x).$$

In Example 6.3, if we change the domain and codomain from set of real numbers $\mathbb{R}$ to set of integers $\mathbb{Z}$, then most students cannot figure out that $y = f(x) = x^3$ does not have an inverse function because they only focus on the formula, and do not know how to show that f is not an onto function from $\mathbb{Z}$ to $\mathbb{Z}$.

Example 6.4. If $y = f(x) = e^x$, then students know that $y = g(x) = \ln x$ is its inverse function. They have no problem to understand the compositions for every positive real number x, we have $f(g(x)) = f(\ln x) = e^{\ln x} = x$, and for every real number z, we have $g(f(z)) = g(e^z) = \ln e^z = z$. So g is the inverse function of f. Also by Horizontal Test, students know that the inverse function of $y = f(x) = e^x$ exists. More generally, for every $a > 1$, $h(x) = a^x$ has an inverse function $k(x) = \log_a x$ because $h(k(x)) = x$ for all $x > 0$ and $k(h(z)) = z$ for every real number z.

Let us compare the inverse function definitions in calculus [20] and abstract algebra [8].

Definition 6.5. (**Stewart**) Let f be a one-to-one function with domain A and range B, then its inverse function f^{-1} has domain B and range A and is defined by $f^{-1}(y) = x$ if and only if $f(x) = y$ for any y in B.

Definition 6.6. (**Herstein**) If f is a bijection from a set A to a set B, then there is a unique bijection f^{-1} from B to A such that $f \circ f^{-1} = Id_B$ and $f^{-1} \circ f = Id_A$. f^{-1} is called the inverse function (mapping) of f.

By Definition 6.5, the cancellation equations can be obtained:

$$f^{-1}(f(x)) = f^{-1}(y) = x$$

for every x in domain A and

$$f(f^{-1}(y)) = f(x) = y$$

for every y in the range B. This is the inverse function definition introduced in the UCSMP (the University of Chicago School Mathematics Project) textbook "Advanced Algebra" [7] which is very helpful for students to learn the abstract and complicated functions in college.

The advantage of Definition 6.5 is that the relationship between the original function and its inverse function is clear. The disadvantage is that it is difficult to use: students need to verify if the function is one-to-one and find out its range, which is not given and difficult to verify for students in general. Taking abstract algebra helps students learn what type of functions have the inverse functions. First, let f be a bijection from a set A to a set B. Define a new function g from B to A as follows. For every y in B, since f is both one-to-one and onto, there is a unique x in A such that $f(x) = y$. We define $g(y) = x$. Then it is not hard for students to see that the two cancellation equations hold. On the other hand, if two cancellation equations hold, then f is a bijection [8]. So the inverse function existence theorem is: the inverse function $g = f^{-1}$ of f exists if and only if the function f is a bijection [8].

After students work on these three examples partially learned in high school and know the construction of the new function g based on the given function f, they can understand the composition process and cancellation equations: We define that a function g from B to A is the inverse of f, denoted by f^{-1} if two cancellation equations hold: $f \circ g = Id_B$ and $g \circ f = Id_A$, where Id_A and Id_B are identity functions: $Id_A : A \to A$, and $Id_B : B \to B$ such that $Id_A(a) = a$ for every a in A and $Id_B(b) = b$ for every b in B.

Our teaching experience with mathematics majors including (pre-service) teachers and community college instructors shows that they understand the above examples and the proof of the existence of the inverse functions: the existence of the inverse function f^{-1} is determined by the necessary and sufficient condition that the function f is a bijection. The most difficult functions for students are trigonometric functions and inverse trigonometric functions. In order to *"Produce an invertible function from a non-invertible function by restricting the domain"* (CCSS.Math.Content.HSF.BF.B.4.d), now we can apply the inverse function existence theorem to trigonometric functions which are difficult for most students in calculus class.

Example 6.7. Let $y = f(x) = \sin x$. Then f is a periodic function with minimum positive period 2π and range $[-1, 1]$. On the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the sine function is strictly increasing therefore one-to-one. If we restrict its domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then sine function is a bijection from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $[-1, 1]$. Its inverse function $f^{-1}(x) = \sin^{-1} x$ exists and the inverse function has domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Example 6.8. Let $y = f(x) = \cos x$, then $\cos(k\pi/2) = 0$ for all integers k. The domain of $\tan x = \frac{\sin x}{\cos x}$ is $x \neq \frac{k\pi}{2}$ and range is $(-\infty, \infty)$. Particularly, in the open interval $(-\pi/2, \pi/2)$, $\tan x$ is a strictly increasing function, so by horizontal test, it is one-to-one and its inverse function $y = \tan^{-1} x$ exists.

7. Discussion and Conclusion

From the authors' teaching experience and the existing literature [2, 3, 4, 5, 13, 17, 25], most universities accept students who are not prepared for college-level mathematics. In the report released by Association of American Colleges and Universities in April, 2016 [2], *"Eighty-six percent of community college students agree or strongly agree that they are academically prepared to succeed at the college where they have enrolled—but 68 percent of students are required take at least some developmental education"* and *"many of them performed well in high school"*. Also *"Most students estimate they will complete their degrees much more quickly than the reality—61 percent of respondents said they expected to complete their academic goals in two years or less, and 76 percent said they were on track to complete their academic goals within their expected time frame. However, only 39 percent of community college students complete a degree or certificate within six years."* A study of the Baltimore City Public Schools class of 2011 found that *"many students with 'A' averages being barred from college-level classes"* [3]. The investigation in the Hechinger Report [3] shows that in the fall of 2014, among 44 states, at least 569,751 students (may not include part time or adult students) at public two-year and four-year campuses were enrolled in remedial classes. Particularly, many students are not ready to learn calculus [4, 5] even though most of them think that they have sufficient knowledge to do well in calculus [2]. So, the students' expectations or self-assessments are different from the reality based on their real skills and faculty's assessment.

Why are so many students not learning the necessary mathematics in high school and middle school [26]? Many factors affect students' learning. According to the report *"Teaching Mathematics in Seven Countries, Results from the TIMSS 1999 Video Study"* released in March 2003 by National Center for Education Statistics [9], there is an important reason: classroom teaching. By comparing teaching methods in three countries, the difference of teaching methods is clear based on geometry lessons in 8th grade classes [21]:

"Germany: Developing advanced learners, challenging content, develop procedures in class. Much attention to content knowledge.

Japan: Structured problem solving, refining of lessons.

United States: Learning terms and practicing procedures. Present definitions of terms and demonstrate procedures. Not interactive, not attending to thinking, not engaging in technique, shallow demonstration of content knowledge."

In the past, many American students were trained to memorize the procedures and do the same type of problems repeatedly without understanding the concepts in depth when they needed to learn how to solve the problems by connecting different concepts, analyzing, reasoning and building a bridge between known methods and the unknown new problems. Because of the gap between the student expectations and the real mathematics skills they learn in high school, based on the STEM data collected by National Science Board [17], in the area of physical, computer, and mathematical sciences, only 43% of freshmen in academic year 2003/04 who had plans to major in these areas were retained. Even though lower retention rate may be caused by many factors, one important reason is that many students do not understand a key mathematical concept: function well and have limited fundamental understanding and reasoning abilities which are essential for learning calculus [4, 5], an important course for all STEM majors. How important is calculus? A MIT applied mathematics professor Daniel Kleitman wrote in his book *"Calculus for Beginners and Artists"* [10]:

"The development of calculus and its applications to physics and engineering is probably the most significant factor in the development of modern science beyond where it was in the days of Archimedes. And this was responsible for the industrial revolution and everything that has followed from it including almost all the major advances of the last few centuries."

Our finding is that some mathematics and mathematics education majors including some graduate students, high school even community college mathematics teachers in our abstract algebra class do not correctly understand the notations, domain and range, composition functions and inverse functions in a profound way. The purpose of studying and understanding these connections is in hopes to improve the way concepts are presented at the high school level and give students a deeper understanding of functions in order to take calculus and other mathematics courses at colleges and universities.

The process of learning mathematics is: students make observations, explore and test new ideas, make mistakes, correct the mistakes and reach conclusions. Before we introduce an abstract concept, it is effective to first relate it to something concrete, geometrical or familiar. We need to connect old knowledge to new concepts and restructure their thinking. We hope to use their familiar functions as examples and the rigorous approach to help students including pre-service and in-service teachers have a strong conceptual deep understanding of functions:

(1) A function as a mapping from a set A to another set B, consists three parts: the rule f, a domain A and a codomain B, where the codomain B may be different from the range of the function.

(2) A one-to-one (or injective) function is a function which does not map two elements in the domain to one element in the range and it satisfies the horizontal test if the domain and range are subsets of the set of real numbers $\mathbb{R}$.

(3) An onto (or surjective) function is a function such that the codomain is equal to its range.

(4) A function is bijective if it is both one-to-one and onto.

(5) A function has an inverse function if and only if it is a bijection.

The high school texts and classroom teaching should be improved in order to prepare students for college courses, particularly, STEM careers. From 1983, the University of Chicago School Mathematics Project (UCSMP) has created educational materials for all students from pre-kindergarten all the way through 12-th grade including *"Advanced Algebra"* [7], *"Functions, Statistics, and Trigonometry"* [11], and *"Precalculus and Discrete Mathematics"* [18]. In these three consecutive textbooks, eight mathematical practices in the Common Core are emphasized, functions are treated as special kinds of relations, the general function concepts including inverse functions are carefully defined, horizontal test theorem for inverse functions is included and trigonometric functions are explored. These books can prepare students for the future study of calculus course in college based on our teaching experience and the research conducted by Thompson [22, 23, 24],

1. *"...the results throughout this report indicate that UCSMP Advanced Algebra (Third Edition) can be an effective textbook to enhance students' learning of important algebraic content in preparation for a future course in precalculus."*

2. *"...the results throughout this report indicate that UCSMP Functions, Statistics, and Trigonometry (FST3) can be an effective textbook to enhance students' learning of important precalculus content while also studying concepts that may prove useful in college courses other than calculus, particularly statistics and probability."*

3. *"...the results throughout this report indicate that UCSMP Precalculus and Discrete Mathematics (Third Edition) can be an effective textbook to enhance students' learning of important precalculus content while also studying concepts that may prove useful in college courses other than calculus."*

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