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SOLVING HIGHER ORDER DIFFERENTIAL EQUATIONS: HIGHER ORDER INTEGRATING FACTOR

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Solving Higher Order Differential Equations: Higher Order Integrating Factor

Synopsis:

This paper aims to show how we can apply integrating factor to higher order differential equations. We will then show second order integrating factor then third order integrating factor then derivation of using integrating factor on an nth order differential equation. Afterwards, we will go through some examples of the methods, and conclude with an analysis of the methods and discussion of future research.

1 Introduction

Ordinary differential equations have a variety of methods that can be used to solve them. These methods include variation of parameters, Euler method, first order integrating factor, and many more. In the previous papers we discussed self-adjoint differential equations and various properties of self-adjoint differential equations [4]. We then were able to show the generalized solutions to self-adjoint differential equations[1]. Through a field in science we applied these findings [5].

This paper aims to show how we can apply integrating factor to higher order differential equations. It is apparent that first order integrating factor can only be used for first order differential equations while other methods are able to solve higher order differential equations. This paper is structured as follows the derivation of second and third order integrating factor using the fact we know the homogeneous solution of the second order differential equation. We will then show the derivation of using integrating factor on an nth order differential equation. Afterwards, we will go through some examples of the methods, and conclude with an analysis of the methods and discussion of future research.

2 Integrating-Factor Approach

2.1 Proposition 1

Given P(x), Q(x), and f(x) are continuous functions and

$$y^{''}(x) + P(x)y^{'}(x) + Q(x)y(x) = f(x) \tag{1}$$

Then

$$y(x) = y_h(x)\int \frac{1}{y_h^2(x)}e^{-\int P(x)dx}\int f(x)y_h(x)e^{\int P(x)dx} + y_h(x)\int \frac{c_2}{y_h^2(x)}e^{-\int P(x)dx} + c_1 y_h$$

Where y_h is the homogeneous solution of 1 and c_2, c_1 are constants.

2.2 Proof

Let

$$y^{''}(x) + P(x)y^{'}(x) + Q(x)y(x) = f(x)$$

Then we can find the homogeneous solution of $y^{''}$(x) + P(x)$y^{'}$(x)+Q(x)y(x) = f(x), where y_h denotes the homogeneous solution.

Now from [2] we have that b = $\frac{-y_h^{'}(x)}{y_h(x)}$ and $u_2(x) = e^{\int P(x)dx}y_h(x)$, we can write the original equation as

$$\begin{aligned}
[u_2(x)(y^{'}(x) + b(x)y(x))]^{'} =& u_2(x)f(x) \\
[e^{\int P(x)dx}y_h(x)(y^{'}(x) - \frac{y_h^{'}(x)}{y_h(x)}y(x))]^{'} =& e^{\int P(x)dx}y_h(x)f(x)
\end{aligned}$$

Integrating both sides we get

$$\begin{aligned}
e^{\int P(x)dx}y_h(x)(y^{'}(x) - \frac{y_h^{'}(x)}{y_h(x)}y(x)) =& \int e^{\int P(x)dx}y_h(x)f(x) + c_2 \\
y^{'}(x) - \frac{y_h^{'}(x)}{y_h(x)}y(x) =& e^{-\int P(x)dx}\frac{1}{y_h(x)}\int e^{\int P(x)dx}y_h(x)f(x)dx + \frac{c_2}{y_h(x)}e^{-\int P(x)dx}
\end{aligned}$$

Now this is a first order differential equation thus we can apply the first order integration factor method.
Let

$$u_1(x) = e^{\int b(x)dx} = e^{\int \frac{-y_h'(x)}{y_h(x)}dx} = \frac{1}{y_h(x)}$$

It follows that

$$(u_1(x)y(x))' = u_1(x)[e^{-\int P(x)dx}\frac{1}{y_h(x)}\int e^{\int P(x)dx}y_h(x)f(x)dx + \frac{c_2}{y_h(x)}e^{-\int P(x)dx}]$$
$$(u_1(x)y(x))' = e^{-\int P(x)dx}\frac{1}{y_h^2(x)}\int e^{\int P(x)dx}y_h(x)f(x)dx + \frac{c_2}{y_h^2(x)}e^{-\int P(x)dx}$$

Integrating both sides we get

$$u_1(x)y(x) = \int e^{-\int P(x)dx}\frac{1}{y_h^2(x)}\int e^{\int P(x)dx}y_h(x)f(x)dx + \int \frac{c_2}{y_h^2(x)}e^{-\int P(x)dx} + c_1$$
$$y(x) = y_h(x)\int e^{-\int P(x)dx}\frac{1}{y_h^2(x)}\int e^{\int P(x)dx}y_h(x)f(x)dx + y_h(x)\int \frac{c_2}{y_h^2(x)}e^{-\int P(x)dx} + c_1 y_h$$

Which completes the proof.

2.3 Proposition 2

Given P(x), Q(x), R(x) , and f(x) are continuous functions

$$y'''(x) + P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = f(x)$$

Then

$$y(x) = y_h(x)\int \frac{1}{y_h^2(x)}e^{-\int \tilde{P}(x)dx}\int y_h(x)e^{\int \tilde{P}(x)dx}\frac{1}{u_3(x)}\int u_3(x)f(x) +$$
$$y_h(x)\int \frac{1}{y_h^2(x)}e^{-\int \tilde{P}(x)dx}\int y_h(x)e^{\int \tilde{P}(x)dx}\frac{c_3}{u_3(x)} + y_h\int \frac{c_2}{y_h^2(x)}e^{-\int \tilde{P}(x)dx} + c_1 y_h(x)$$

Where $u_3(x)$ is the homogeneous solution of u(x) and $y_h(x)$ is the homogeneous to the second order equation and c_3, c_2, c_1 are constants and $\tilde{P}(x) = \frac{P(x)u_3(x) - u_3'(x)}{u_3(x)}$

2.4 Proof

From [1] we have that

$$u'''(x) - P(x)u''(x) + [Q(x) - 2P'(x)]u'(x) + [Q'(x) - P''(x) - R(x)]u(x) = 0$$

We can find the homogeneous solution for u which will be denoted as $u_3(x)$.
Now

$$b(x) = P(x) - \frac{u_3'(x)}{u_3(x)} = \frac{P(x)u_3(x) - u_3'(x)}{u_3(x)}$$
$$a(x) = \frac{Q(x)u_3^2(x) - P'(x)u_3^2(x) + u_3(x)u_3''(x) - P(x)u_3'(x)u_3(x)}{u_3^2(x)}$$

Then

$$[u_3(x)(y^{''}(x) + b(x)y^{'}(x) + a(x)y(x))]^{'} = u_3(x)f(x)$$

Integrating both sides results in,

$$u_3(x)(y^{''}(x) + b(x)y^{'}(x) + a(x)y(x)) = \int u_3(x)f(x) + c_3$$

Since a(x) and b(x) are continuous functions we get that $\tilde{Q}$(x) = a(x) and $\tilde{P}$(x) = b(x). Then it follows that,

$$y^{''}(x) + \tilde{P}(x)y^{'}(x) + \tilde{Q}(x)y(x) = \frac{1}{u_3(x)}\int u_3(x)f(x) + \frac{c_3}{u_3(x)}$$

We can find a homogeneous solution for this second order differential equation, which will be denoted as $y_h(x)$. We can now apply the second order integration factor method.
Let

$$\tilde{b}(x) = -\frac{y_h^{'}(x)}{y_h(x)}$$
$$u_2(x) = e^{\int \tilde{P}(x)dx}y_h(x)$$

Then

$$[u_2(x)(y^{'}(x) + \tilde{b}(x)y(x))]^{'} = u_2(x)(\frac{1}{u_3(x)}\int u_3(x)f(x) + \frac{c_3}{u_3(x)})$$

We integrate both sides and get,

$$u_2(x)(y^{'}(x) + \tilde{b}(x)y(x)) = \int u_2(x)(\frac{1}{u_3(x)}\int u_3(x)f(x) + \frac{c_3}{u_3(x)} + c_2$$
$$(y^{'}(x) + \tilde{b}(x)y(x)) = \frac{1}{u_2(x)}(\int u_2(x)(\frac{1}{u_3(x)}\int u_3(x)f(x) + \frac{c_3}{u_3(x)}) + c_2)$$

From here, we can apply the first order integrating factor method.
Let

$$u_1(x) = e^{\int \tilde{b}(x)dx} = \frac{1}{y_h(x)}$$

Thus,

$$[u_1(x)y(x)]^{'} = u_1(x)[\frac{1}{u_2(x)}(\int u_2(x)(\frac{1}{u_3(x)}\int u_3(x)f(x) + \frac{c_3}{u_3(x)}) + c_2)]$$

Integrating both sides results in,

$$u_1(x)y(x) = \int u_1(x)[\frac{1}{u_2(x)}(\int u_2(x)(\frac{1}{u_3(x)}\int u_3(x)f(x) + \frac{c_3}{u_3(x)}) + c_2)] + c_1$$
$$y(x) = \frac{1}{u_1(x)}[\int u_1(x)[\frac{1}{u_2(x)}(\int u_2(x)(\frac{1}{u_3(x)}\int u_3(x)f(x) + \frac{c_3}{u_3}) + c_2)] + c_1]$$

Now we can distribute our u_is, this results in

$$y = \frac{1}{u_1(x)}\int\frac{u_1(x)}{u2(x)}\int\frac{u_2(x)}{u_3(x)}\int u_3(x)f(x) + \frac{1}{u_1(x)}\int\frac{u_1(x)}{u_2(x)}\int\frac{u_2(x)}{u_3(x)}c_3 + \frac{1}{u_1(x)}\int\frac{u_1(x)}{u_2(x)}c_2 + \frac{c_1}{u_1(x)}$$

Now $u_1(x) = \frac{1}{y_h(x)}$ and $u_2(x) = e^{\int \tilde{P}(x)dx}y_h(x)$, so it follows,

$$\text{y(x)} = y_h(x)\int\frac{1}{y_h^2(x)}e^{-\int\tilde{P}(x)dx}\int y_h(x)e^{\int\tilde{P}(x)dx}\frac{1}{u_3(x)}\int u_3f(x)dx +$$
$$y_h(x)\int\frac{1}{y_h^2(x)}e^{-\int\tilde{P}(x)dx}\int y_h(x)e^{\int\tilde{P}(x)dx}\frac{c_3}{u_3(x)}dx + y_h(x)\int\frac{c_2}{y_h^2(x)}e^{-\int\tilde{P}(x)dx} + c_1y_h$$

Which completes the proof.

2.5 Proposition 3

Given an nth order general linear ODE,

$$y^n(x) + P_n(x)y^{n-1}(x) + \ldots + P_2(x)y^{'}(x) + P_1(x)y(x) = f(x)$$

Then the integrating factor equation is the following;

$$0 = (u_nP_2)^{'} - (u_nP_3)^{''} + (u_nP_4)^{'''} + \ldots + (u_nP_{n-1})^{n-2}(-1)^{n-1} + (u_nP_n - u_n^{'})^{n-1}(-1)^n - u_nP_1$$

2.6 Proof

Given an nth order general linear ODE,

$$y^n(x) + P_n(x)y^{n-1}(x) + \ldots + P_2(x)y^{'}(x) + P_1(x)y(x) = f(x)$$

we can express this this Nth order equation as:

$$[\text{u(x)}(\text{y}^{n-1}\text{(x)} + b_{n-1}\text{(x)y}^{n-2}\text{(x)} + \ldots + b_2\text{(x)y}^{'}\text{(x)} + b_1\text{(x)y(x)}]^{'} = \text{u(x)f(x)}$$

Furthermore from [2]

$$\begin{aligned}
u(x)P_n(x) &= u^{'}(x) + u(x)b_{n-1}(x)\\
u(x)P_{n-1}(x) &= u^{'}(x)b_{n-1}(x) + u(x)(b_{n-1}(x))^{'} + u(x)b_{n-2}(x)\\
&\vdots\\
u(x)P_2(x) &= u^{'}(x)b_2(x) + u(x)(b_2(x))^{'} + u(x)b_1(x)\\
u(x)P_1(x) &= u(x)b_1^{'}(x) + u^{'}(x)b_1(x)
\end{aligned}$$

Now, we solve for $u(x)b_{n-1}$, this results

$$u(x)b_{n-1}(x) = u(x)P_n(x) - u^{'}(x)$$

From here we see that,

$$\begin{aligned}
u(x)P_{n-1}(x) &= u^{'}(x)b_{n-1}(x) + u(x)(b_{n-1}(x))^{'} + u(x)b_{n-2}(x)\\
u(x)P_{n-1}(x) &= [u(x)b_{n-1}(x)]^{'} + u(x)b_{n-2}(x)
\end{aligned}$$

Now we can plug $u(x)b_{n-1}$ into this, and solve for $u(x)b_{n-2}(x)$

$$\begin{aligned}
u(x)b_{n-2} &= u(x)P_{n-1}(x) - [u(x)b_{n-1}(x)]^{'}\\
u(x)b_{n-2}(x) &= u(x)P_{n-1}(x) - [u(x)P_n(x) - u^{'}(x)]'
\end{aligned}$$

We then repeat the process until we get to $u(x)P_1(x)$, which results in

$$u(x)P_1(x) = [u(x)b_1(x)]'$$

Expanding out $u(x)b_1(x)$, we get

$$\begin{gathered}
u(x)P_1(x) = (u_n(x)P_2(x))^{'} - (u_n(x)P_3(x))^{''} + (u_n(x)P_4(x))^{'''} + \ldots +\\
(u_n(x)P_{n-1}(x))^{n-2}(-1)^{n-1} + (u_n(x)P_n(x) - u_n^{'}(x))^{n-1}(-1)^n
\end{gathered}$$

2.7 Theorem

Consider an n[th] order linear ODE,

$$y^n(x) + P_n(x)y^{n-1}(x) + \ldots + P_2(x)y'(x) + P_1(x)y(x) = f(x)$$

Where $P_1(x) \ldots P_n(x)$ and f(x) are continuous functions and where n $\in$ **N**. Then

$$y(x) = \frac{1}{u_1(x)} \int \frac{u_1(x)}{u_2(x)} \int \frac{u_2(x)}{u_3(x)} \cdots \int \frac{u_{n-1}(x)}{u_n(x)} \int u_n(x)f(x) + \frac{1}{u_1(x)} \int \frac{u_1(x)}{u_2(x)} \int \frac{u_2(x)}{u_3(x)} \cdots \int \frac{u_{n-1}(x)}{u_n(x)} k_n + \ldots + \frac{1}{u_1(x)} \int \frac{u_1(x)}{u_2(x)} k_2 + \frac{k_1}{u_1(x)}$$

Where $u_1(x) \ldots u_n$ are continuous integrating factors and $k_1 \ldots k_n$ are constants, where n $\in$ **N**.

2.8 Proof

Now given

$$y^n(x) + P_n(x)y^{n-1}(x) + \ldots + P_2(x)y'(x) + P_1(x)y(x) = f(x)$$

we can express this this N[th] order equation as:

$$[u(x)(y^{n-1}(x) + b_{n-1}(x)y^{n-2}(x) + \ldots + b_2(x)y'(x) + b_1(x)y(x)]' = u(x)f(x)$$

Expanding this out we get,

$$u'(x)(y^{n-1}(x) + b_{n-1}(x)y^{n-2}(x) + \ldots + b_2(x)y'(x) + b_1(x)y(x)) + u(x)(y^n(x) + b'_{n-1}(x)y^{n-2}(x) + b_{n-1}(x)y^{n-1}(x) + \ldots + b'_1(x)y(x) + b_1(x)y'(x) = u(x)f(x)$$

Furthermore,

$$u(x)y^n(x) + y^{n-1}(x)(u'(x) + u(x)b_{n-1}(x)) + \ldots + y(x)(u(x)b'_1(x) + u(x)'b_1(x)) = u(x)f(x)$$

By proposition 3, we can find the equation for the integrating factor, which results in,

$$u(x)P_1(x) = (u_n(x)P_2(x))' - (u_n(x)P_3(x))'' + (u_n(x)P_4(x))''' + \ldots + (u_n(x)P_{n-1}(x))^{n-2}(-1)^{n-1} + (u_n(x)P_n(x))^{n-1}(-1)^n$$

From this we can find the solution for this equation we will denote it as $u_n(x)$. Furthermore, we have the integration factor equation differential equation with solution $u_n(x)$, then by the previous part we get

$$[u_n(x)(y^{n-1}(x) + b_{n-1}(x)y^{n-2}(x) + \ldots + b_2(x)y'(x) + b_1(x)y(x)]' = u_n(x)f(x)$$

Integrating on both sides we get

$$y^{n-1}(x) + b_{n-1}(x)y^{n-2}(x) + \ldots + b_2(x)y'(x) + b_1(x)y(x) = \frac{1}{u_n(x)} \int u_n(x)f(x) + \frac{k_n}{u_n(x)}$$

Now let g(x) = $\frac{1}{u_n(x)} \int u_n(x)f(x) + \frac{k_n}{un(x)}$

$$y^{n-1}(x) + b_{n-1}(x)y^{n-2}(x) + \ldots + b_2(x)y'(x) + b_1(x)y(x) = g(x)$$

Now we can repeat this process again. We can find the integrating factor equation $u_{n-1}(x)$ in terms of $b_1(x), \ldots, b_{n-1}(x)$. This let's find a solution of $u_{n-1}(x)$.
This results in,

$$[u_{n-1}\text{(x)}(\text{y}^{n-2}\text{(x)} + \tilde{b}_{n-2}\text{(x)}\text{y}^{n-3}\text{(x)} + \ldots + \tilde{b}_2\text{(x)}\text{y}'\text{(x)} + \tilde{b}_1\text{(x)}\text{y(x)}]' = u_{n-1}\text{(x)g(x)}$$

and

$$\text{y}^{n-2}\text{(x)} + \tilde{b}_{n-2}\text{(x)}\text{y}^{n-3}\text{(x)} + \ldots + \tilde{b}_2\text{(x)}\text{y}'\text{(x)} + \tilde{b}_1\text{(x)y(x)} = \frac{1}{u_{n-1}(x)}\int u_{n-1}\text{(x)g(x)} + \frac{k_{n-1}}{u_{n-1}(x)}$$

From here, we can use this process to reduce the order of y, until we are able to use first order integrating factor, which results in this expression.

$$(u_1\text{(x)y(x)})' = \frac{u_1(x)}{u_2(x)}\int\frac{u_2(x)}{u_3(x)}\cdots\int\frac{u_{n-1}(x)}{u_n(x)}\int u_n(x)f(x) + \frac{u_1(x)}{u_2(x)}\int\frac{u_2(x)}{u_3(x)}\cdots\int\frac{u_{n-1}(x)}{u_n(x)}k_n + \ldots + \frac{u_1(x)}{u_2(x)}k_2$$

Integrating and solving for y we are able to get,

$$\text{y(x)} = \frac{1}{u_1(x)}\int\frac{u_1(x)}{u_2(x)}\int\frac{u_2(x)}{u_3(x)}\cdots\int\frac{u_{n-1}(x)}{u_n(x)}\int u_n(x)f(x) + \frac{1}{u_1(x)}\int\frac{u_1(x)}{u_2(x)}\int\frac{u_2(x)}{u_3(x)}\cdots\int\frac{u_{n-1}(x)}{u_n(x)}k_n + \ldots + \frac{1}{u_1(x)}\int\frac{u_1(x)}{u_2(x)}k_2 + \frac{k_1(x)}{u_1(x)}$$

Which completes the proof.

3 Examples

3.1 Example 1

We will show how to apply proposition 1 to a problem.
Given the second order linear differential equation,

$$y'' + y = x \tag{2}$$

We have that P(x) = 0, Q(X) = 1 and f(x) = x, now a homogeneous solution of this equation is sin(x) thus $y_h(x)$ = sin(x).
Thus by proposition 1 we have

$$y(x) = y_h(x)\int\frac{1}{y_h^2(x)}e^{-\int P(x)dx}\int f(x)y_h(x)e^{\int P(x)dx} + y_h(x)\int\frac{c_2}{y_h^2(x)}e^{-\int P(x)dx} + c_1y_h$$

Plugging in our values we get,

$$y(x) = sin(x)\int\frac{1}{sin(x)^2}e^{-\int 0dx}\int x * sin(x)e^{\int 0dx} + sin(x)\int\frac{c_2}{(sin(x))^2}e^{-\int 0dx} + c_1sin(x)$$

We then solve for the integrals.

$$sin(x)\int\frac{c_2e^c}{(sin(x))^2} = c_2^*sinx\frac{cos(x)}{sin(x)} = c_2cos(x)$$

The other integral is a bit more involved,

$$sin(x)\int\frac{1}{sin(x)^2}e^{-\int 0dx}\int x * sin(x)e^{\int 0dx}$$

$$sin(x)\int\frac{1}{sin(x)^2}\int x * sin(x)$$

$$sin(x)\int\frac{1}{sin(x)^2}[-xcos(x) + sin(x)]$$

$$sin(x)[\int\frac{1}{sin(x)} - \int\frac{xcos(x)}{sin^2(x)}]$$

$$x - sin(x)ln(\frac{sin(x)}{cos(x)+1}) + sin(x)ln(\frac{sin(x)}{cos(x)+1})$$

$$x$$

Which results in,

$$y(x) = x + c_2 cos(x) + c_1 sin(x)$$

3.2 Example 2

Given the third order linear differential equation,

$$y'''(x) + (\frac{1}{x} + x)y''(x) + (3 - \frac{1}{x^2})y'(x) = 0$$

We have that P(x) = $(\frac{1}{x} + x)$, Q(x) = $-\frac{1}{x^2}$, R(x) = 0. From proposition 2 we can make the respective integrating factor equation, it follows that,

$$u^{'''}(x) - P(x)u^{''}(x) + [Q(x) - 2P^{'}(x)]u^{'}(x) + [Q^{'}(x) - P^{''}(x) - R(x)]u(x) = 0$$
$$u^{'''}(x) - (\frac{1}{x} + x)u^{''}(x) + (1 + \frac{1}{x^2})u^{'}(x) = 0$$

We can clearly see that a homogeneous solution is a constant, thus $u_3 = k$ where k is a constant. Before we can apply the rest of corollary 3, we have to find y_h, we can do this by using third order integrating factor as discussed in [1]. Using u_3, we can see that the second order equation is

$$y'' + (\frac{1}{x} + x)y' + 2y = 0$$

Solving for the homogeneous equation we see that $y_h = e^{-\frac{x^2}{2}}$. From [1], we can use this homogeneous solution to find the first order equation, since b = $-\frac{y_h'}{y_h}$.

$$\frac{(e^{-\frac{x^2}{2}})'}{e^{-\frac{x^2}{2}}} = x$$

Thus the first order equation is

$$y' + xy = 0$$

We can then solve for u_2

$$\frac{(u_2)'}{u_2} = P_2 - b_1$$
$$\frac{(u_2)'}{u_2} = \frac{1}{x} + x - x$$
$$u_2 = x$$

Now $u_1 = \frac{1}{y_h} = e^{\frac{x^2}{2}}$, now that we have found all of our integrating factors we can apply the theorem to find the solution to the third order equation.

$$y = \frac{1}{u_1}\int\frac{u_1}{u_2}\int\frac{u_2}{u_3}\int u_3 f(x) + \frac{1}{u_1}\int\frac{u_1}{u_2}\int\frac{u_2}{u_3}c_3 + \frac{1}{u_1}\int\frac{u_1}{u_2}c_2 + \frac{1}{u_1}c_1$$

Now plugging in our u's and f(x) = 0, we get

$$y = \frac{1}{e^{\frac{x^2}{2}}}\int\frac{e^{\frac{x^2}{2}}}{x}\int\frac{x}{k}c_3 + \frac{1}{e^{\frac{x^2}{2}}}\int\frac{e^{\frac{x^2}{2}}}{x}c_2 + \frac{1}{e^{\frac{x^2}{2}}}c_1$$

We can simplify one of the integrals, which results in,

$$y = \frac{c_3}{2} + e^{-\frac{x^2}{2}}\int\frac{e^{\frac{x^2}{2}}}{x}c_2 + e^{-\frac{x^2}{2}}c_1$$

3.3 Example 3

Using proposition 3 we will find the equation for third order integrating factor equations. Given a third order linear differential equation,

$$y''' + P_3 y'' + P_2 y' + P_1 y = f(x)$$

We observe that for n = 3,

$$0 = u_3^{'''}(x) - (u_3 P_3)^{''} + (u_3 P_2)^{'} - u_3 P_1$$

If we expand this equation we arrive to,

$$0 = u_3^{'''} + u_3^{'}(P_2 - 2P_3^{'}) + u_3(P_2^{'} - P_3^{''} - P_1)$$

This equation is the same as the one we found in [2].

3.4 Example 4

We will use the theorem to show the general method of fourth order equations.
Given

$$y''''(x) + P_4(x)y'''(x) + P_3(x)y''(x) + P_2(x)y'(x) + P_1(x)y(x) = f(x)$$

with $P_i(x)$ and f(x) being continuous, then by theorem we have

$$y(x) = \frac{1}{u_1(x)}\int \frac{u_1(x)}{u_2(x)}\int \frac{u_2(x)}{u_3(x)}\int \frac{u_3(x)}{u_4(x)}\int u_4 f(x) + \frac{1}{u_1(x)}\int \frac{u_1(x)}{u_2(x)}\int \frac{u_2(x)}{u_3(x)}\int \frac{u_3(x)}{u_4(x)} c_4 + \frac{1}{u_1(x)}\int \frac{u_1(x)}{u_2(x)}\int \frac{u_2(x)}{u_3(x)} c_3 + \frac{1}{u_1(x)}\int \frac{u_1(x)}{u_2(x)} c_2 + \frac{1}{u_1(x)} c_1$$

where $u_i s$ are integrating factors and $c_i s$ are constants

3.5 Example 5

We can use the derivation to find the solution to a given 4$^{\text{th}}$ order equation.
Given

$$y^{''''} + \frac{2}{x}y^{'''} + (25 - 25x^2)y^{''} + (\frac{30}{x} - 100x)y^{'} + -50y = f(x) \tag{3}$$

We can apply the lemma to 3 find u_4, thus we get $u_4 = xe^{\frac{-5}{2}x^2}$. This then lets us apply integrating factor to reduce the order of the equation, resulting in

$$y^{'''} + (\frac{1}{x} + 5x)y^{''} + 20y^{'} + \frac{10}{x}y$$

We can then apply the lemma again, and we $u_3 = x$, which then lets us reduce the order again,

$$y^{''} + 5xy^{'} + 10y$$

We can repeat this process to get $u_2(x) = x$ and $u_1(x) = \frac{1}{x}e^{\frac{5}{2}x^2}$. From here we can plug in all of our $u_i's(x)$ into the solution of the previous problem. It follows,

$$y = xe^{-\frac{5}{2}x^2}\int \frac{e^{\frac{5}{2}x^2}}{x^2}\int 1 \int e^{\frac{5}{2}x^2}\int xe^{-\frac{5}{2}x^2} f(x) + xe^{-\frac{5}{2}x^2}\int \frac{e^{\frac{5}{2}x^2}}{x^2}\int 1 \int e^{\frac{5}{2}x^2} c_4 + xe^{-\frac{5}{2}x^2}\int \frac{e^{\frac{5}{2}x^2}}{x^2}\int c_3 + xe^{-\frac{5}{2}x^2}\int \frac{e^{\frac{5}{2}x^2}}{x^2} c_2 + xe^{-\frac{5}{2}x^2} c_1$$

This is the solution for the fourth order differential equation.

3.6 Example 6

This section will be a discussion of a question from [6] which showed the usage of Ricatti equations [1]. To be more exact it is the Kortewge-De Vries equation.
Given

$$\frac{\partial}{\partial t}w - 6w\frac{\partial w}{\partial x} + \frac{\partial^3}{\partial x^3}w = 0$$

Through the transformations X = x - ct, we result in the third ordered equation

$$-c\frac{\partial}{\partial X}f - 6f\frac{\partial}{\partial X}f + \frac{\partial^3}{\partial X^3}f = 0$$

Furthermore, the resulting Ricatti equations is

$$-\frac{\partial}{\partial X}(\frac{\frac{\partial}{\partial X}\bar{f}}{\bar{f}}) + (\frac{\frac{\partial}{\partial X}\bar{f}}{\bar{f}})^2 - \frac{c}{4} = 0$$

Both of the third order and Ricatti equation are related to each other, and they both provide insight to the original equation. Although [6] provides a solution to this, we use can higher order integration factor with respect to X to solve the third order equation. This would give us the solution the Ricatti equation, and to the original equation.

4 Conclusion

4.1 Conclusion

In conclusion, this paper established how to apply integrating factor for higher order ordinary linear differential equations. We were able to show the procedure to use integrating factor for second and third order equation given that we know the homogeneous solution of the equation. This can be further expanded on as the theorem gives a more general process to finding the solutions given the order of the equation. This is extremely useful as we can apply integrating factor on any order of equation. The problem arises as we have to find the integrating factors as well as solve the integrals. Although with further research we will be able to reduce the process of finding the integrating factors.

5 Future Research

There is various parts of integrating factor that we are trying to focus on. One of them is being able to use the theorem on the n^{th} order. The theorem is very difficult to work with since we have to find n integrating factors. Although that is when you are working with the theorem from n^{th} order to get the solutions, but with the same theorem we can start from any order linear differential equation and reach an n^{th} order linear differential equation with all of its solutions. The restriction is that we have to pick the integrating factors

5.1 Example 7

given the first order equation,

$$y' + 2xy = 0$$

Let us pick $u_2 = x$, then we can construct the second order equation

$$y'' + (\frac{1}{x} + 2x)y' + 4y = 0$$

Now first order integrating factor gives us $u_1 = e^{x^2}$, which then lets us apply the theorem, so we have have solutions for the 2nd order equation.

$$y = e^{-x^2} \int \frac{e^{x^2}}{x} c_2 + e^{-x^2} c_1$$

The other area we are trying to work on, is real life examples of applying integrating factor through the use of Riccati equations. One real life example we will try to focus on is Riccati equations in the medical field.

5.2 Example 8

From [3] we are given the regular Ricatti equation.

$$y'(x) = p(x)y^2(x) + q(x)y + r(x)$$

Now given a field such as cardiology, as stated in [7] we can model heart rate through the use of Ricatti equations. To be more exact ”the dynamics of electrical signals that control contractions of the heart. Specifically an action potential which is an electrical signal V(t) that travels through the signals of the cardiac muscles, which stimulates the contractions of hate muscles at a given time t” [7]. Now we can model action potential through the equation.

$$V'(t) = pV^2(t) + qV(t) + r \tag{4}$$

Now p,q,r are real valued constants that can help us determine properties of the treated patient. These values can be determined, but are not necessarily needed for the the use of this paper. The more relevant part is that from [1] and [2] given a first order Ricatti equation we find a respective second order equation. Furthermore, with the second order equation we can find the solutions for 4.

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