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REGIME SWITCHING PROCESS IN A MULTIPLE DECREMENTS MODEL

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Regime Switching Model in a Multiple Decrements

Abstract

We propose a modification of multiple decrement models used in actuarial science, biostatistics, and other applications of survival models. Multiple decrements are used in actuarial science to mathematically model future lifespan of a person in a situation when the cause of death affects the amount of death benefit paid upon death. We supplement such a basic model by allowing a mortality change through random regime switching in one or more decrements. We show how regime switching in one decrement affects observed mortality in another decrement. We simulate such a regime switching process in a multiple decrement model and show how it can be calibrated to real world data.

Key words: multiple decrements, regime switching, actuarial science.

1. Introduction

In the year 1900, the two most common causes of death in the U.S. were tuberculosis and polio. In the year 2000, the two most common causes were cancer and cardiovascular diseases. Does this mean that in 1900 Americans did not suffer as much from cancer and cardiovascular diseases? Not so, they simply did not live long enough to be affected by those two diseases, as they died at earlier ages from diseases such as tuberculosis and polio. When we study human longevity, we must understand that curing one disease does not lead to a nirvana with no sickness, but rather it helps improve longevity, but in that longer lifespan, humans will be more affected by other diseases and other causes of death, such as car accidents, as elimination of one disease naturally leads to greater exposure to other possible diseases and causes of death.

In actuarial science, which is the study of mathematical models used in insurance applications, this issue is studied with the concept of *multiple decrements.* Decrements are defined as various causes of death, or methods of departure from the insured group (for example, departure of an employee from an employment group that we are modeling the cost of an occupational pension plan, by termination of employment). Multiple decrement models are used in actuarial science, and in other areas, especially biostatistics, to describe survival of humans or other species in the context of multiple causes of death competing to end the life of a person or another entity studied.

2. Basic multiple decrements model

In the basic multiple decrements model in actuarial science, we consider a person purchasing some form of life insurance, aged *x,* denoted by (*x*). Then *T* is the future lifetime of (*x*), a continuous random variable. We also have a random variable $K = [T]$, the greatest integer function applied to *T,* which is the integer portion of *T.* For example, if *T* = 15.25, then *K* = 15. *J*: mode of decrement at death (departure from the group under consideration). This is a discrete random variable. Then we consider a random variable *J,* a discrete one, whose values correspond to specific causes of death, or departure from the group of insured people. *J* is assumed to have only a finite number of possible values. Effectively, we are now studying the joint distribution of these random variables *T* and *J* (or *K* and *J*). The following basic notation is used:
$l_x^{(\tau)}$: observed number of people in the group (cohort) at age *x.*
$d_x^{(j)}$: observed number of people departing the group (cohort) between ages *x* and *x* + 1 due to cause *j*.
$q_x^{(j)} = \frac{d_x^{(j)}}{l_x^{(\tau)}}$: observed probability that (*x*) departs the group in the next year due to cause *j,* with other decrements also after (*x*) (i.e., with competition from other causes).
$q_x^{(\tau)} = \sum_{j=1}^{m} q_x^{(j)}$: observed probability of departure, regardless of cause, when all causes compete.
We will introduce additional notation used as needed in the body of the paper.

Let us note that in actuarial science, this issue is of importance when different causes of death, or different methods of departure from the group, result in different costs of insurance benefits. For example, an employee in a qualified pension plan who leaves employment before his/her benefits are vested, may receive no pension benefits at all, while an employee who leaves employment later, may receive fractional benefits, or even full benefits if the employee works all the way to the retirement age.

3. Adding regime switching to the picture

We study the effect of a regime switching process in a multiple decrements model where there are two causes of deaths:

- Cause 1: whose force of mortality changes according to a regime switching process with two states: mild and severe. In other words, the mathematical structure of mortality switches randomly between these two states. We stipulate that the force of mortality in the mild state will always be lower than the force of mortality in the severe state. *Force of mortality* at age *x,* denoted by μ_x, is defined as the ratio of the probability density function at that age divided by the probability of survival to that age $s(x) = \Pr(X > x)$.
- Cause 2: whose force of mortality is not related to a regime switching regime.

In our study we suppose that cause 1 is similar to the flu that can be mild flu or severe flu as well as cause B is related to any other diseases. This is a very basic simplified model, which we use to present the idea, and illustrate with numerical values.

The aim of this note is to study the probability of person age *t* years old dies in one year due to a specific cause, $q_t^{(j)}$ where $j = 1$ or $j = 2$, and to see if there is a relationship between the probability of death and the state of the regime switching process.

The aim is to study the probability of a person of year t to die in one year due to cause *j* and see what happens when there is a change in the force of mortality due to a different state in the regime switching process. For most of the analysis, we assume constant force of mortality for each decrement (with the regime for cause 1 switching from constant to another).

The numerical procedure for simulation is as follows:

(i) Estimate a vector that defines for each time $t = 0, 1, \ldots, T$, if we are in state 1 (mild flu) or state 2 (severe flu).

2. Create for each time $t = 0, 1, \ldots, T$, the life table that contains only the number of people alive at time t, l_t, and the number of people that died between year t and $t + 1$, denoted by d_t.

3. Compute for each time $t = 0, 1, \ldots, T$, the probability of a person at time t to die in one year due to cause 1, $q_t^{(1)}$, or due to cause 2, $q_t^{(2)}$.

Regime switching: Basic model

We have two regimes, i.e., $s_t = 1$, or $s_t = 2$, where 1 is the regime where the flu is mild and 2 is the regime where the flu is severe. We assume that the transition matrix is constant and it does not change over time:

$$\begin{bmatrix} P & 1-P \\ 1-Q & Q \end{bmatrix},$$

where

$$P = \Pr\left(s_t = 1 \middle| s_{t-1} = 1 \right), \qquad Q = \Pr\left(s_t = 2 \middle| s_{t-1} = 2 \right).$$

Based on this, we compute the unconditional probability of being in state 1 as

$$\Pr(s = 1) = \frac{1-Q}{2-P-Q}.$$

In the numerical model that follows, we assume that the regime switch can happen only at the end of each year, $t = 1, 2, \ldots, T$, and s_t indicates which state the process is in at time *t.*

Given a uniform random variable x_t with $t = 0, 1, 2, \ldots, T$, we have

• When $t = 0$,

$$s_t = s_0 = \begin{cases} 1 & \text{if } x_0 < \Pr(s = 1), \\ 2 & \text{otherwise.} \end{cases}$$

• When $t > 0$,

$$s_t = \begin{cases} 1 \text{ if } \left(s_{t-1} = 1 \text{ and } x_t < P \right) \text{ or } \left(s_{t-1} = 2 \text{ and } x_t < 1 - Q \right), \\ 2 \text{ otherwise.} \end{cases}$$

Let's assume that $P = 0.90$ and $Q = 0.80$ and T=100 years: Simulation results for this case are provided in Figure 1.

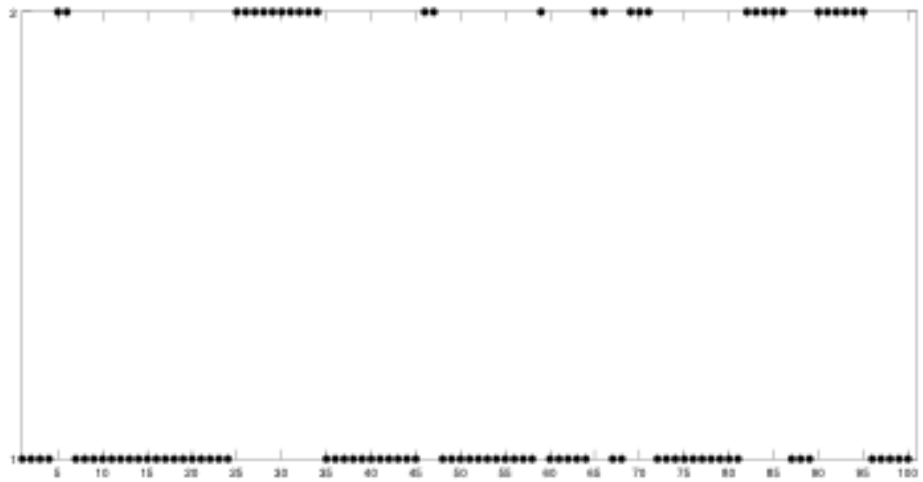

Figure 1: Simulation of regimes

For the current model, we assume constant force of mortality. Of course, that assumption can be modified in future research. To further elaborate on the simulation process, we show the computation of l_t and d_t, defined as:

• l_t is the size of the population studied at time t, with possible values of t being $t = 0, 1, 2, \ldots, T$.

• d_t is the number of deaths observed in the population between time t and $t + 1$.

We assume that people can die from two causes:

• Flu: In regime 1 has a mild force of mortality, $\mu_{\text{flu},1}$, and in regime 2 has a severe force of mortality, $\mu_{\text{flu},2}$.

• Other diseases: Whose force of mortality does not change in different regimes and is always $\mu_{\text{OD},1}$.

For construction of the mortality table, we note that

$$l_{t+1} = l_t \cdot e^{-\int_0^1 \sum_{\text{all } j} \mu_s^{(j)} ds}.$$

Here are the steps in the process of creation of the mortality tabe:

1. Assume the parameters $l_0 = 1000$, $\mu_{\text{flu},1} = 0.01$, $\mu_{\text{flu},2} = 0.05$, and $\mu_{\text{OD},1} = 0.01$.
2. For $0 \le t < T, l_{t+1} = l_t \cdot e^{-\left(\mu_{\text{flu},s_t} + \mu_{\text{OD}}\right)}$, $d_t = l_t - l_{t+1}$, where $s_t = 1$ or 2, and s_t defines the regime at time t, and hence it defines the force of mortality applicable for the flu.

Figure 2 represents the results of our computation, and shows the graphs of l_t and d_t, and we note that gray shadowed area represents the times when the regime switches to severe flu.

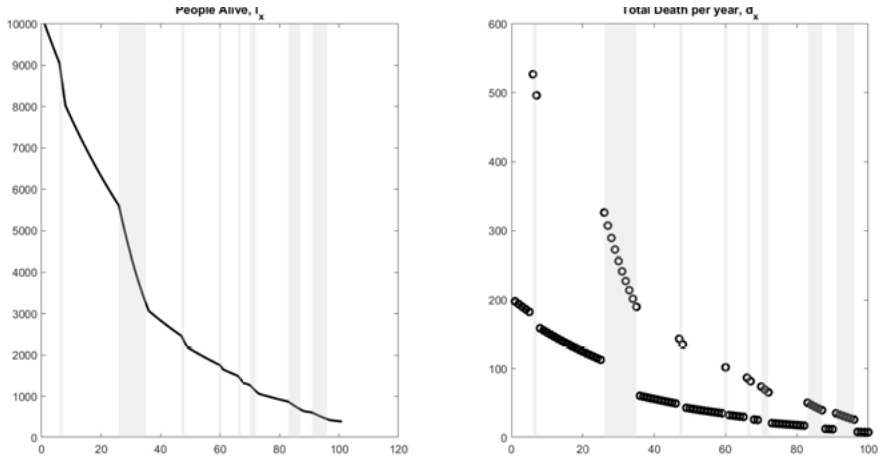

Figure 2: Graphs of of l_t and d_t,

4. Computing probability of death and survival probability

As we have the number of people that are alive and dead at each *t,* we compute the probability of survival of one year for a person of age *t* as $p_t = \frac{l_{t+1}}{l_t}$, and the probability of a person of t year to die between year *t* and *t* + 1 as $q_t = 1 - p_t = \frac{d_t}{l_t}$. Under constant force assumption we are using in this work we also have: $\frac{q_t^{(j)}}{q_t^{(\tau)}} = \frac{\mu_t^{(j)}}{\mu_t^{(\tau)}}$. We will, from now on, omit the indicator of all decrements operating, the letter *τ,* so that we can write: $\frac{q_t^{(j)}}{q_t} = \frac{\mu_t^{(j)}}{\mu_t}$. And more specifically, we compute:

$$q_t^{(1)} = \begin{cases} \frac{\mu_{flu,1}}{\mu_{flu,1}+\mu_{OD}} q_t & \text{if } s_t = 1 \\ \frac{\mu_{flu,2}}{\mu_{flu,2}+\mu_{OD}} q_t & \text{if } s_t = 2 \end{cases}$$

$$q_t^{(2)} = \begin{cases} \frac{\mu_{OD}}{\mu_{flu,1}+\mu_{OD}} q_t & \text{if } s_t = 1 \\ \frac{\mu_{OD}}{\mu_{flu,2}+\mu_{OD}} q_t & \text{if } s_t = 2 \end{cases}$$

Note that the total force of mortality is the sum of the force of mortality due to flu and the force of mortality due to other diseases. This is illustrated in Figure 3 below. We see that the probability are always the same over time, that is because we have a constant force of mortality over time. What is interesting to note is that when the flu is severe (regime 2, in gray area in the plot), the probability of dying due to other diseases slightly diminishes. This makes sense as when the regime is 2, we compute:

$$q_t^{(2)} = \frac{\mu_{OD}}{\mu_{flu,2}+\mu_{OD}} q_t \text{ .}$$

In this case the denominator increases (compared to the one in regime 1) which means that the force of mortality OD has less power over all mortality.

Another approach could be, to compute the probability of death for cause j as:

$$ {}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)} \mu^{(j)}(x+s)ds $$

$$ q_x^{(j)} = \int_0^1 e^{-\int_0^s \sum_j \mu^{(j)}(x+y)dy} \mu^{(j)}(x+s)ds $$

$$ q_x^{(j)} = \mu^{(j)} \int_0^1 e^{-\sum_j \mu^{(j)} s} ds $$

$$ q_x^{(j)} = \mu^{(j)} \frac{1 - e^{-\sum_j \mu^{(j)}}}{\sum_j \mu^{(j)}} $$

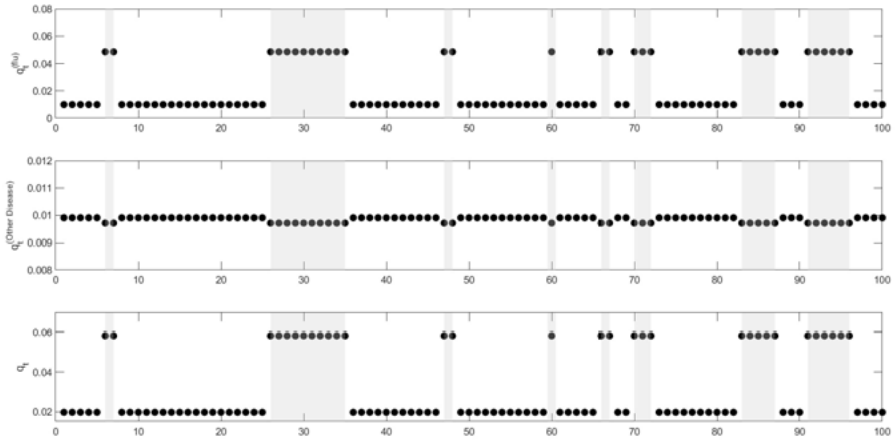

Figure 3: Probability of dying under regime switching

In general, we demonstrate that

$$ {}_tq_x^{(\mathrm{OD})}(\text{State 1}) > {}_tq_x^{(\mathrm{OD})}(\text{State 2}) $$

$$ \int_o^t {}_sp_x^{(\tau)} \mu^{OD}(x+s)ds\,[\text{State 1}] > \int_o^t {}_sp_x^{(\tau)} \mu^{OD}(x+s)ds\,[\text{State 2}] $$

$$ \frac{d}{dt}\int_o^t {}_sp_x^{(\tau)} \mu^{OD}(x+s)ds\,[\text{State 1}] > \frac{d}{dt}\int_o^t {}_sp_x^{(\tau)} \mu^{OD}(x+s)ds\,[\text{State 2}] $$

$$ {}_tp_x^{(\tau)} \mu^{OD}(x+t)\,[\text{State 1}] > {}_tp_x^{(\tau)} \mu^{OD}(x+t)\,[\text{State 2}] $$

$$ {}_tp_x^{(\tau)}\,[\text{State 1}] > {}_tp_x^{(\tau)}\,[\text{State 2}] $$

5. Using Real Data

We obtain the data from the public server of the U.S. Center for Disease Control. We downloaded the following data:

• Number of people alive for years 1999 through 2020,
• Number of people who died each year,
• Number of people dead due to the flu.

We can see the total death due to the flu are really a small quantity compared to the total death. Then, we compute the probability of death that are summarized in graph and we see that the probability to die due to the flu is so small that does not result in any changes in the probability of dying of all the other diseases. We do not see that the death due to flu can have an impact on the total death. We note that in 2018, where the flu death are the highest in the last 20 years, the people that died from the flu are equal to the 0.39% of the total death whereas in 2001 they are equal to 0.0106% of the total death.

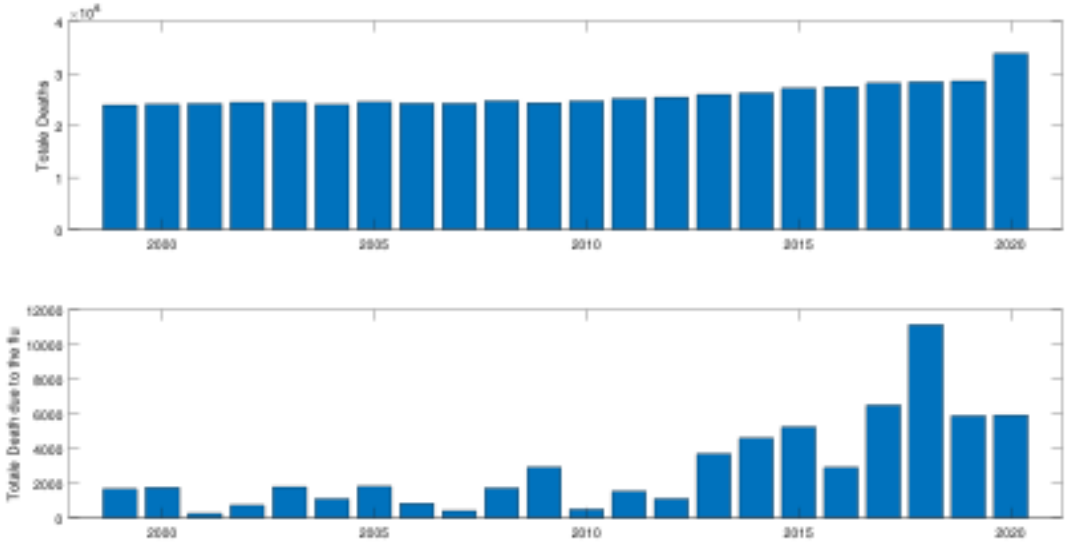

Figure 4: Total deaths and deaths due to flu

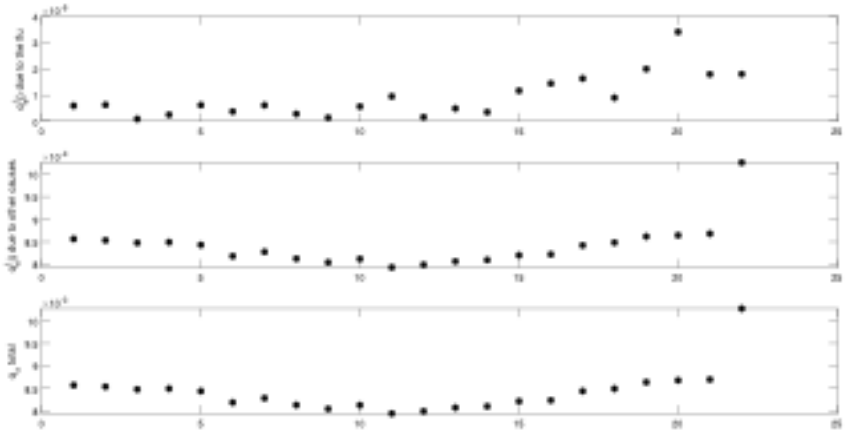

Figure 5: Probabilities of dying

For this reason, we decided to compare the impact of flu death on the total death caused by Influenza and Pneumonia Death (from now on called Pneumonia

Death) that contains as a subcase the Flu death. From Figure 6 we can see that the flu death has more impact on the Pneumonia Death. For example, in 2018, the flu death were 18.88% of the Pneumonia Death and in the 2001 they were equal to 0.41% of them. We compared the di erent probability of death and we can see that when the death probability from flu is higher (for example around 2017-2020), the probability of dying due to all the other causes included in Pneumonia and Influenza is smaller. We can also see the same relation in the scatter plot where we can see that an increase of the total death to the flu would mean a decrease of the total death due to Pneumonia and Influenza (excluding the flu). In this graph, we can see that we can assume that there are two regimes where the cut off is the q_t^{flu} is equal to 0.00075%. If the probability of dying due to the flu is bigger, we are in a scenario where the flu is severe which leads us to have a smaller probability of dying due to other causes.

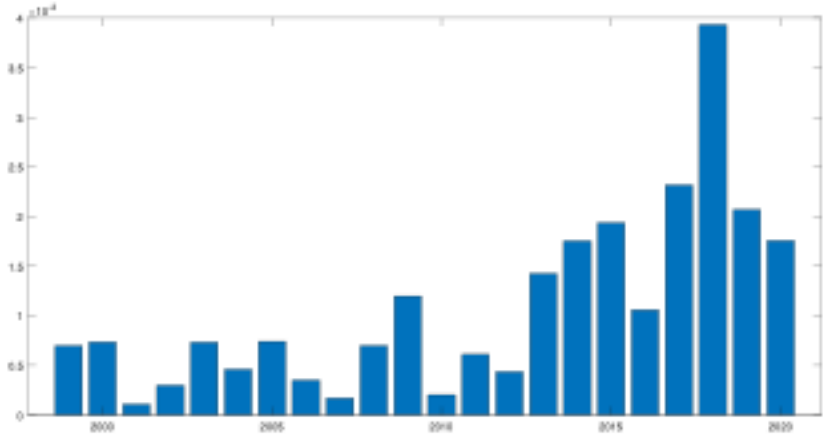

Figure 6: Percentage of people who died due to flu

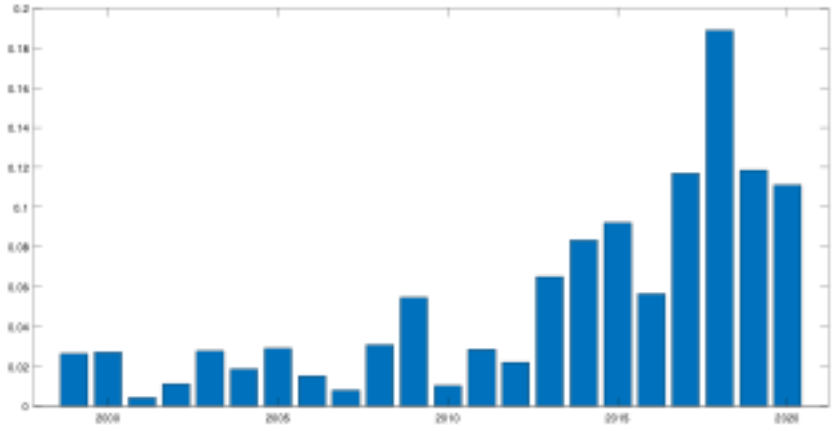

Figure 7: Percentage of combined Influenza and Pneumonia deaths due to flu

Finally, Figure 8 shows the scatterplot of the patters of mortality.

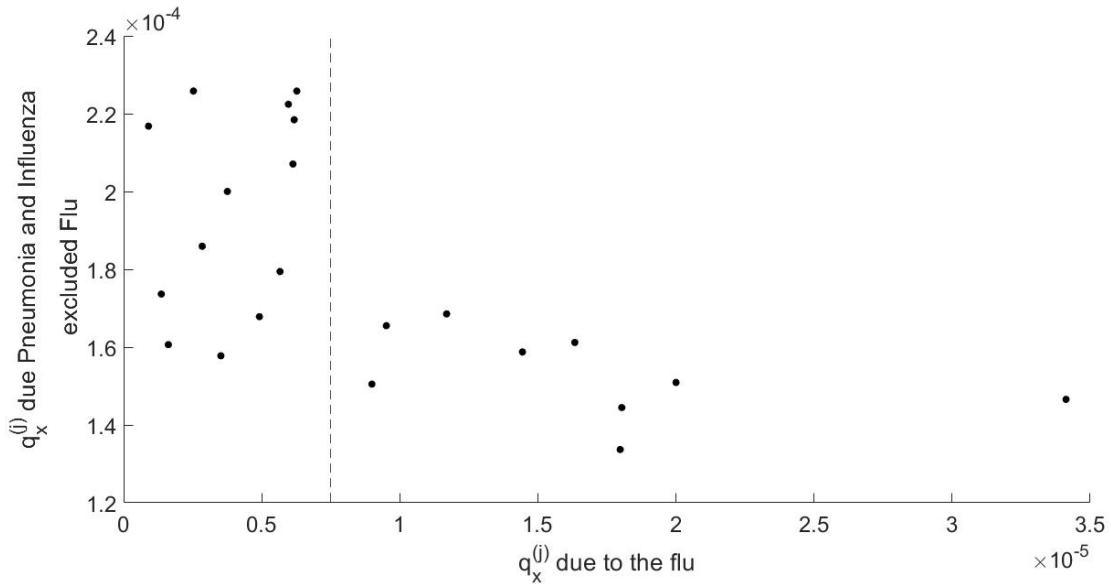

Figure 8: Scatterplot

Conclusions

In this paper we propose an enhancement to traditional multiple decrements models used in modality survival and mortality in life insurance, life annuities and pensions, as well as in biostatistics. We propose an addition of a regime switching to account for sudden changes in mortality pattens in one of the decrements, we explain the underlying model, show the consequences of this enhanced model, and perform some simulation. We also show how the approach can be supported by real data.

We hope this enhancement can be of possible value for actuarial applications, and be further developed in future research.

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