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BI-HARMONIC MAPS' EXISTANCE AND p-BALANCED ENERGY AND CONNECTIONS TO HARMONIC MAPS

WU, LINA
YUAN, HONG
DEPARTMENT OF MATHEMATICS
BOROUGH OF MANHATTAN COMMUNITY COLLEGE
THE CITY UNIVERSITY OF NEW YORK
NEW YORK, NEW YORK
LIU, JIA
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF WEST FLORIDA
PENSACOLA, FLORIDA
LI, YE
DEPARTMENT OF MATHEMATICAL SCIENCES
WILLIAM RAINEY HARPER COLLEGE
PALATINE, ILLINOIS

Prof. Lina Wu
Dr. Hong Yuan
Department of Mathematics
Borough of Manhattan Community College
The City University of New York
New York, New York
Prof. Jia Liu
Department of Mathematics and Statistics
University of West Florida
Pensacola, Florida
Dr. Ye Li
Department of Mathematical Sciences
William Rainey Harper College
Palatine, Illinois

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Synopsis:

We discover bi-harmonic maps' existence, energy growth, and connections to harmonic maps. A nontrivial bi-harmonic map's existence in a unit sphere is proved. We investigate the p-balanced energy as energy technical breakthroughs to break the constraints of the existing q-energy technique in detecting q-energy towards boundlessness. We study the connections between bi-harmonic maps and harmonic maps by discovering a rigid dichotomy between infinite energy and finite constant energy.

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Abstract

We discover bi-harmonic maps' existence, energy growth, and connections to harmonic maps. First, we prove the existence of a nontrivial bi-harmonic map in a unit sphere. Second, we investigate the p-balanced energy growth of bi-harmonic maps. As the most important energy technical breakthroughs, we propose an innovative energy algorithm called p-balanced energy technique to break the constraints of the existing q-energy technique in detecting q-energy growth towards boundlessness. The disadvantage of the finite q -energy technique in L^q spaces is not effective i n dealing with infinite q- energy in Non-L^q spaces. Third, we study the connections between bi-harmonic maps and harmonic maps by employing our p-balanced energy technique to discover a rigid dichotomy between infinite e nergy a nd fi nite co nstant energy. We demonstrate the theoretical verification for our research findings. The research implication on $\mathcal{A}$-harmonic theory in physics is also discussed.

Keywords: tension field, 2-tension field, Laplacian operator, harmonic maps, bi-harmonic maps, q-energy growth, p-balanced energy growth, Convergence Tests for Infinite Series, Divergence Theorem, Integration by Parts, Cauchy-Schwarz Inequality

1. Introduction and Background

Since James Eells and Luc Lemainre [3] raised questions about bi-harmonic maps in 1983, many mathematicians have conducted research on bi-harmonic maps' existence, energy, stability on manifolds with various geometric structures as well as its applications in physics, engineering, and medicine [11]. In 1986, G. Jiang [6] obtained the knowledge of the first and second variational formulas in the study of bi-harmonic stability. In 2007, Changyou Wang [15] established the existence of a solution to the heat flow of extrinsic bi-harmonic maps where both the domain and the target space are compact Riemannian manifolds without boundary. In 2015, S. Montaldo and A. Ratto [13] constructed proper bi-harmonic sub-manifolds into various types of ellipsoids. In 2018, Ahmed Mohammed Cherif and Mustapha Djaa [1] applied the variation method to study h-energy of bi-harmonic maps and discovered the connections between bi-harmonic maps and harmonic maps with potential on complete manifolds. In 2022, Ye-Lin Ou [14] used the variation method to compute the normal stability index of bi-harmonic hyper-surfaces in Euclidean spheres and proved the nonexistence of unstable proper bi-harmonic hyper-surfaces in Euclidean spaces or hyperbolic spaces.

In this paper, we focus on bi-harmonic maps' existence, p-balanced energy growth, and connections to harmonic maps. First, we prove the existence of a nontrivial bi-harmonic map in a unit sphere. Second, we investigate the p-balanced energy growth of bi-harmonic maps. As the most important energy technical breakthroughs, we propose an innovative energy algorithm called p-balanced energy technique to break the constraints of the existing q-energy technique in detecting q-energy growth towards boundlessness. The disadvantage of the finite q-energy technique in L^q spaces is not effective in dealing with infinite q-energy in Non-L^q spaces. Third, we study the connections between bi-harmonic maps and harmonic maps by employing our p-balanced energy technique to explore a rigid dichotomy between infinite energy and finite constant energy. We demonstrate the theoretical verification for our research findings. The research implication on $\mathcal{A}$-harmonic theory in physics is also discussed at the end.

2. Preliminary

Here we give definitions of p-balanced energy growth including 5 cases. We also give definitions of harmonic maps and bi-harmonic maps.

2.1. Definition of p-Balanced Energy Growth

We assume that M is a complete non-compact m-dimensional Riemannian manifold and $B(x_0; r)$ (or $B(r)$) is the geodesic ball of radius r centered at a point $x_0 \in M$. A function f on M is said to be with p-balanced growth provided f has one of the following "p-finite, p-mild, p-obtuse, p-moderate, and p-small" growth where $p > 1$. Otherwise, f is said to be with p-imbalanced growth [16].

Definition 1. *A function f has p-finite growth (or, simply, is p-finite) if there exists $x_0 \in M$ such that*

$$\liminf_{r\to\infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q dv < \infty$$

and has p-infinite growth (or, simply, is p-infinite) otherwise.

Definition 2. *A function f has p-mild growth (or, simply, is p-mild) if there exist $x_0 \in M$, and a strictly increasing sequence of $\{r_j\}_0^\infty$ going to infinity, such that for every $l_0 > 0$, we have:*

$$\sum_{j=l_0}^{\infty} \Big(\frac{(r_{j+1} - r_j)^p}{\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)} |f|^q dv}\Big)^{\frac{1}{p-1}} = \infty$$

and has p-severe growth (or, simply, is p-severe) otherwise.

Definition 3. *A function f has p-obtuse growth (or, simply, is p-obtuse) if there exists $x_0 \in M$ such that for every $a > 0$, we have:*

$$\int_a^\infty \left(\frac{1}{\int_{\partial B(x_0;r)} |f|^q dv}\right)^{\frac{1}{p-1}} dr = \infty$$

and has p-acute growth (or, simply, is p-acute) otherwise.

Definition 4. *A function f has p-moderate growth (or, simply, is p-moderate) if there exist $x_0 \in M$, and $F(r) \in \mathcal{F}$, such that*

$$\limsup_{r\to\infty} \frac{1}{r^p F^{p-1}(r)} \int_{B(x_0;r)} |f|^q dv < \infty$$

and has p-immoderate growth (or, simply, is p-immoderate) otherwise, where

$$\mathcal{F} = \{F : [a, \infty) \to (0, \infty) | \int_a^\infty \frac{dr}{rF(r)} = \infty\}$$

for some $a \geq 0$. Notice that the functions in $\mathcal{F}$ are not necessarily monotone.

Definition 5. *A function f has p-small growth (or, simply, is p-small) if there exists $x_0 \in M$, such that for every $a > 0$, we have:*

$$\int_a^\infty \left(\frac{r}{\int_{B(x_0;r)} |f|^q dv}\right)^{\frac{1}{p-1}} dr = \infty$$

and has p-large growth (or, simply, is p-large) otherwise.

The above definitions of "p-finite, p-mild, p-obtuse, p-moderate, p-small" and their counter-parts "p-infinite, p-severe, p-acute, p-immoderate, p-large" growth depend on q, and q will be specified in the context in which the definitions are used [16].

2.2. Definitions of Harmonic Maps and Bi-Harmonic Maps

Definition 6. *A harmonic map $u : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds can be viewed as a critical point of the Energy functional: $E(u) = \frac{1}{2}\int_M \|du\|^2 v_g$. A harmonic map u is characterized by the vanishing of the first tension field, that is, $0 = \tau(u) = trace\overline{\nabla} du = \sum_{i=1}^m (\overline{\nabla}_{e_i} du)(e_i) = \sum_{i=1}^m \overline{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i) = \sum_{i=1}^m \overline{\nabla}_{e_i} du(e_i)$ where $\{e_i\}$ is a local orthonormal frame field on M such that $\nabla_{e_i} e_i = 0$.*

In 1981, J. Eells and L. Lemaire [3] made a conjecture. Conjecture: If $u : (M^m, g) \to (N^n, h)$ is a k-harmonic map, then u is a critical point of the k-Energy functional: $E_k(u) = \frac{1}{2}\int_M \|(d+d^*)^k u\|^2 v_g$. In particular, a 2-harmonic map is called a bi-harmonic map.

Definition 7. *A bi-harmonic map $u : (M^m, g) \to (N^n, h)$ can be viewed as a critical point of the 2-Energy functional $E_2(u) = \frac{1}{2}\int_M \|(d + d^*)^2 u\|^2 v_g$, where d and d^* are the exterior differentiation and the co-differentiation on vector bundle. A bi-harmonic map u is characterized by the vanishing of the bi-tension field, that is, $0 = \tau_2(u) = -\overline{\nabla} * \overline{\nabla}\tau(u) + \sum_{i=1}^m R^N(du(e_i), \tau(u))du(e_i)$ where $\{e_i\}$ is a local orthonormal frame field on M. A rough Laplacian is defined as $\Delta = \Delta^u = -\sum_{i=1}^m \overline{\nabla}_{e_i} * \overline{\nabla}_{e_i} = trace_g(\nabla^u_{e_i}\nabla^u_{e_i} - \nabla^u_{\nabla_{e_i}e_i}) = trace_g(\nabla^u_{e_i}\nabla^u_{e_i})$ where $\{e_i\}$ is a local orthonormal frame field on M such that $\nabla_{e_i}e_i = 0$.*

3. Our Research Findings including Theorems and Proofs

In this section, we discover our research findings on a bi-harmonic map's existence, its p-balanced growth, and its connections to a harmonic map. We verify the theoretical justification in proof.

3.1. The Existence Theorem of A Nontrivial Bi-Harmonic Map in A Unit Sphere

Theorem 1. *Let $S^{m+1}(1)$ be a unit sphere in a Euclidean space $\mathbb{R}^{m+2}$. We use a lower dimensional Euclidean space $\mathbb{R}^{m+1}$ to horizontally cut $S^{m+1}(1)$ and the intersection part is denoted by M^m. Then there exists an inclusion map $u : M^m \longrightarrow S^{m+1}(1)$, which is a nontrivial bi-harmonic map.*

Proof. Here the map $u : M^m \longrightarrow S^{m+1}$ is an inclusion map. $\forall x \in M^m$, Suppose $\{e_1, \ldots, e_m\}$ is an o.n.b. in TM^m and $\{e_1, \ldots, e_m, \gamma\}$ is an o.n.b. in TS^{m+1}, and $\{e_1, \ldots, e_m, \gamma, x\}$ is an o.n.b. in R^{m+2}, where $\gamma = \frac{\vec{n} - <\vec{n},x>x}{\|\vec{n} - <\vec{n},x>x\|} = \frac{\vec{n} - <\vec{n},x>x}{\sqrt{<\vec{n} - <\vec{n},x>x, \vec{n} - <\vec{n},x>x>}} = \frac{\vec{n} - <\vec{n},x>x}{\sqrt{1 - <\vec{n},x>^2}} = \frac{\vec{n} - x\|\vec{n}\|\|x\|\cos\phi}{\sqrt{1 - (\|\vec{n}\|\|x\|\cos\phi)^2}} = \frac{\vec{n} - x\cos\phi}{\sin\phi}$ where $\vec{n}$ is a unit vector orthogonal ($\|\vec{n}\| = \sqrt{<\vec{n},\vec{n}>} = 1$) to M^m, x is a unit vector ($\|x\| = \sqrt{<x,x>} = 1$) on a unit Sphere S^{m+1}, and ϕ is an angle between a

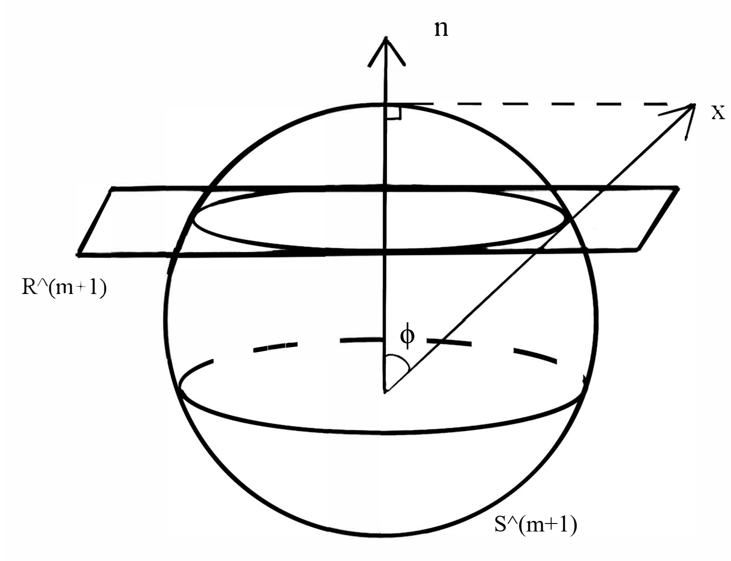

Figure 1: Figure of Bi-Harmonic in Unit Sphere

unit vector x and a unit vector $\vec{n}$ (See Figure 1) By the definition of $\tau(u)$, we calculate:

$$\begin{aligned}
\tau(u) &= \sum_{i=1}^{m} \overline{\nabla}_{e_i}^{S^{m+1}} du(e_i) \\
&= \sum_{i=1}^{m} \overline{\nabla}_{e_i}^{S^{m+1}} e_i \\
&= \sum_{i=1}^{m} < \overline{\nabla}_{e_i}^{S^{m+1}} \gamma, e_i > \gamma \\
&= \sum_{i=1}^{m} < \overline{\nabla}_{e_i}^{R^{m+2}} (\frac{\vec{n} - x\cos\phi}{\sin\phi}), e_i > \gamma \\
&= \sum_{i=1}^{m} \frac{-\cos\phi}{\sin\phi} < e_i, e_i > \gamma \\
&= \frac{-m\cos\phi}{\sin\phi}\gamma \\
&= \frac{-m\cos\phi}{\sin\phi}(\frac{\vec{n} - x\cos\phi}{\sin\phi}) \\
&= \frac{mx(\cos\phi)^2 - m\vec{n}\cos\phi}{(\sin\phi)^2}
\end{aligned}$$

where we use the following facts

(i) $\gamma = \frac{\vec{n}- < \vec{n}, x > x}{\|\vec{n}- < \vec{n}, x > x\|} = \frac{\vec{n}- < \vec{n}, x > x}{\sqrt{< \vec{n}- < \vec{n}, x > x, \vec{n}- < \vec{n}, x > x >}}$

$$= \frac{\vec{n}- < \vec{n}, x > x}{\sqrt{1- < \vec{n}, x >^2}} = \frac{\vec{n} - x\cos\phi}{\sqrt{1-(\cos\phi)^2}} = \frac{\vec{n} - x\cos\phi}{\sin\phi},$$

(ii) $\overline{\nabla}^{R^{m+2}}_{e_i}\vec{n} = 0,$

(iii) $\overline{\nabla}^{R^{m+2}}_{e_i} x = e_i,$

(iv) $< \overline{\nabla}^{R^{m+2}}_{e_i} x, e_i >=< e_i, e_i >= 1,$

(v) $< \overline{\nabla}^{R^{m+2}}_{e_i}\vec{n}, e_i >=< 0, e_i >= 0.$

By the definition of $\tau_2(u)$, we have:

$$\begin{aligned}
\tau_2(u) &= -\overline{\nabla} * \overline{\nabla}\tau(u) + \sum_{i=1}^{m} R^{S^{m+1}}(du(e_i), \tau(u))du(e_i) \\
&= -\sum_{i=1}^{m}\overline{\nabla}_{e_i} * \overline{\nabla}_{e_i}\tau(u) + m\tau(u) \\
&= m\frac{\cos^2\phi}{\sin^2\phi}(-\tau(u)) + m\tau(u) \\
&= m\tau(u)\big(1 - \frac{\cos^2\phi}{\sin^2\phi}\big)
\end{aligned}$$

where we use the following facts

(i) $R^{S^{m+1}}(du(e_i), \tau(u))du(e_i) = m\tau(u), \quad i = 1, \ldots, m;$

(ii) The sectional curvature of a unit sphere $S^{m+1}(1)$ is equal to 1;

(iii) $\overline{\nabla}_{e_i}\tau(u) = \overline{\nabla}_{e_i}\frac{mx(\cos\phi)^2 - m\vec{n}\cos\phi}{(\sin\phi)^2}$

$$= \frac{m(\cos\phi)^2}{(\sin\phi)^2}\overline{\nabla}_{e_i}x - \frac{m\cos\phi}{(\sin\phi)^2}\overline{\nabla}_{e_i}\vec{n}$$

$$= \frac{m(\cos\phi)^2}{(\sin\phi)^2}e_i;$$

(iv) $-\sum_{i=1}^{m}\overline{\nabla}_{e_i} * \overline{\nabla}_{e_i}\tau(u) = \frac{m(\cos\phi)^2}{(\sin\phi)^2}(-\sum_{i=1}^{m}\overline{\nabla}^{S^{m+1}}_{e_i} e_i) = \frac{m(\cos\phi)^2}{(\sin\phi)^2}(-\tau(u)).$

Here, if $\frac{(\cos\phi)^2}{(\sin\phi)^2} = 1$ (that is, $|\cos\phi| = |\sin\phi|$) when $\phi = \frac{\pi}{4}$ or $\phi = \frac{3\pi}{4}$, we get $\tau_2(u) = 0$ and $|\tau(u)| = |\frac{-m\cos\phi}{\sin\phi}\gamma| = m > 0$ since $|\cos\phi| = |\sin\phi|$ and γ

is a unit vector, which indicates $\tau(u) \neq 0$. Therefore, we obtain a non-trivial bi-harmonic map in a unit sphere since $\tau_2(u) = 0$ and $\tau(u) \neq 0$. $\square$

Theorem 2. *Let $S^n(1)$ be a unit sphere in a Euclidean space $\mathbb{R}^{n+1}$. We use a lower dimensional Euclidean space $\mathbb{R}^{m+1}$ to horizontally cut $S^n(1)$ and the intersection part is denoted by M^m. Then there exists an inclusion map $u : M^m \longrightarrow S^n(1)$, which is a nontrivial bi-harmonic map.*

Proof. Here the map $u : M \longrightarrow S^n$ is an inclusion map. $\forall x \in M^m$, Suppose $\{e_1, \ldots, e_m\}$ is an o.n.b. in TM^m and $\{e_1, \ldots, e_m, \gamma_1, \ldots, \gamma_{n-m}\}$ is an o.n.b. in TS^n, and $\{e_1, \ldots, e_m, \gamma_1, \ldots, \gamma_{n-m}, x\}$ is an o.n.b. in R^{n+1}, where $\gamma_k = \frac{\vec{n}_k - <\vec{n}_k, x> x}{\|\vec{n}_k - <\vec{n}_k, x> x\|} = \frac{\vec{n}_k - <\vec{n}_k, x> x}{\sqrt{<\vec{n}_k - <\vec{n}_k, x> x, \vec{n}_k - <\vec{n}_k, x> x>}} = \frac{\vec{n}_k - <\vec{n}_k, x> x}{\sqrt{1 - <\vec{n}_k, x>^2}} = \frac{\vec{n}_k - x\cos\phi_k}{\sqrt{1 - (\cos\phi_k)^2}} = \frac{\vec{n}_k - x\cos\phi_k}{\sin\phi_k}, 1 \leq k \leq n-m$ where $\vec{n}_k$ is a unit vector orthogonal to M^m and x is a unit vector on S^n and ϕ_k is an angle between a unit vector x and a unit vector $\vec{n}_k, 1 \leq k \leq n-m$.

Our calculations of $\tau(u)$ and $\tau_2(u)$ are listed below:

$$\begin{aligned}
\tau(u) &= \sum_{i=1}^{m} \overline{\nabla}_{e_i}^{S^n} du(e_i) \\
&= \sum_{k=1}^{n-m} \sum_{i=1}^{m} \overline{\nabla}_{e_i}^{S^n} e_i \\
&= \sum_{k=1}^{n-m} \sum_{i=1}^{m} < \overline{\nabla}_{e_i}^{S^n} \gamma_k, e_i > \gamma_k \\
&= \sum_{k=1}^{n-m} \sum_{i=1}^{m} < \overline{\nabla}_{e_i}^{R^{n+1}} \left(\frac{\vec{n}_k - x\cos\phi_k}{\sin\phi_k}\right), e_i > \gamma_k \\
&= \sum_{k=1}^{n-m} \frac{-\cos\phi_k}{\sin\phi_k} \sum_{i=1}^{m} < e_i, e_i > \gamma_k \\
&= \sum_{k=1}^{n-m} \frac{-m\cos\phi_k}{\sin\phi_k} \gamma_k \\
&= \sum_{k=1}^{n-m} \frac{-m\cos\phi_k}{\sin\phi_k} \left(\frac{\vec{n}_k - x\cos\phi_k}{\sin\phi_k}\right) \\
&= m \sum_{k=1}^{n-m} \frac{x(\cos\phi_k)^2 - \vec{n}_k \cos\phi_k}{(\sin\phi_k)^2}
\end{aligned}$$

where we use the following facts

(i) $\gamma_k = \dfrac{\vec{n}_k - <\vec{n}_k, x> x}{\|\vec{n}_k - <\vec{n}_k, x> x\|} = \dfrac{\vec{n}_k - <\vec{n}_k, x> x}{\sqrt{<\vec{n}_k - <\vec{n}_k, x> x, \vec{n}_k - <\vec{n}_k, x> x>}}$

$$= \frac{\vec{n}_k - <\vec{n}_k, x> x}{\sqrt{1 - <\vec{n}_k, x>^2}} = \frac{\vec{n}_k - x\cos\phi_k}{\sqrt{1-(\cos\phi_k)^2}} = \frac{\vec{n}_k - x\cos\phi_k}{\sin\phi_k}$$

(ii) $\overline{\nabla}^{R^{n+1}}_{e_i} \vec{n}_k = 0$

(iii) $\overline{\nabla}^{R^{n+1}}_{e_i} x = e_i$

(iv) $<\overline{\nabla}^{R^{n+1}}_{e_i} x, e_i> = <e_i, e_i> = 1$

(v) $<\overline{\nabla}^{R^{n+1}}_{e_i} \vec{n}_k, e_i> = <0, e_i> = 0$

$$\begin{aligned}
\tau_2(u) &= -\overline{\nabla} * \overline{\nabla}\tau(u) + \sum_{i=1}^{m} R^{S^n}(du(e_i), \tau(u))du(e_i) \\
&= -\sum_{i=1}^{m} \overline{\nabla}_{e_i} * \overline{\nabla}_{e_i}\tau(u) + m\tau(u) \\
&= -m\sum_{k=1}^{n-m} \frac{\cos^2\phi_k}{\sin^2\phi_k}\tau(u) + m\tau(u) \\
&= m\tau(u)\big(1 - \sum_{k=1}^{n-m} \frac{\cos^2\phi_k}{\sin^2\phi_k}\big)
\end{aligned}$$

where we use the facts

$$\begin{aligned}
\overline{\nabla}_{e_i}\tau(u) &= \overline{\nabla}_{e_i}\Big(m\sum_{k=1}^{n-m} \frac{x\cos^2\phi_k - \vec{n}_k\cos\phi_k}{\sin^2\phi_k}\Big) \\
&= m\sum_{k=1}^{n-m} \frac{\cos^2\phi_k}{\sin^2\phi_k}\overline{\nabla}_{e_i}x - \frac{\cos\phi_k}{\sin^2\phi_k}\overline{\nabla}_{e_i}\vec{n}_k \\
&= m\sum_{k=1}^{n-m} \frac{\cos^2\phi_k}{\sin^2\phi_k}e_i.
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{m} \overline{\nabla}_{e_i} * \overline{\nabla}_{e_i}\tau(u) &= \sum_{i=1}^{m} \overline{\nabla}_{e_i}\Big(m\sum_{k=1}^{n-m} \frac{\cos^2\phi_k}{\sin^2\phi_k}e_i\Big) \\
&= m\sum_{k=1}^{n-m} \frac{\cos^2\phi_k}{\sin^2\phi_k}\Big(\sum_{i=1}^{m} \overline{\nabla}^{S^n}_{e_i} e_i\Big) \\
&= m\sum_{k=1}^{n-m} \frac{\cos^2\phi_k}{\sin^2\phi_k}\tau(u).
\end{aligned}$$

Thus, u is a nontrivial bi-harmonic map if and only if $\tau_2(u) = 0$ and $\tau(u) \neq 0$ when $\sum_{k=1}^{n-m} \frac{\cos^2 \phi_k}{\sin^2 \phi_k} = 1$. □

Based on the above two examples, we can get the following theorem:

Theorem 3. *In a unit sphere $S^n(1)$, $\forall 1 \leq m < n$, there exists a nontrivial bi-harmonic map $u : M^m \to S^n(1)$, where M^m is a sub-manifold in this unit sphere.*

3.2. The p-Balanced Energy Growth for Bi-Harmonic Maps

Lemma 1. *Suppose a C^2 function $f \geq 0$ on a Riemannian manifold M satisfies the following two conditions:*

1. $\Delta f \geq 0$

2. $\liminf_{r\to\infty} \frac{\int_{B(r)} f^2 dv}{r^2} < \infty$

then f must be a constant function, that is, $f = C$.

Proof. Since $\Delta = div(\nabla)$ and our assumptions of $\Delta f \geq 0$ and $f \geq 0$, we have

$$\begin{aligned} div(\nabla(f^2)) &= div(2f\nabla f) \\ &= 2 < \nabla f, \nabla f > +2f div(\nabla f) \\ &= 2|\nabla f|^2 + 2f\Delta f \\ &\geq 2|\nabla f|^2 \end{aligned}$$

We define a rotationally symmetric Lipschitz continuous function $\xi(x) = \xi(x_0, s, t)$ and a constant $C_1 > 0$ (independent of x_0, s, t) for $0 < s < t$ on M with the following properties:

(i) $\xi = 1$ on $B(x_0, s)$

(ii) $\xi = 0$ off $B(x_0, t)$

(iii) $0 < \xi < 1$ on $B(x_0, t) \backslash B(x_0, s)$

(iv) $|\nabla \xi| \leq \frac{C_1}{t-s}$ a.e. on M.

$$
\begin{aligned}
0 &= \int_{B(x_0,t)} \frac{1}{2} div(\xi^2 \nabla(f^2)) dv & (1)\\
&= \int_{B(x_0,t)\setminus B(x_0,s)} 2f\xi < \nabla\xi, \nabla f > dv + \int_{B(x_0,t)} \frac{1}{2}\xi^2 div(\nabla(f^2)) & (2)\\
&\geq \int_{B(x_0,t)\setminus B(x_0,s)} 2f\xi \langle \nabla\xi, \nabla f\rangle dv + \int_{B(x_0,t)} \xi^2 |\nabla f|^2 dv & (3)
\end{aligned}
$$

where we apply the Divergence Theorem in (1), the derivative product rule in (2), and $div(\nabla(f^2)) \geq 2|\nabla f|^2$ in (3).

$$
\begin{aligned}
&\int_{B(x_0,t)} \xi^2 |\nabla f|^2 dv \\
&\leq -\int_{B(x_0,t)\setminus B(x_0,s)} 2f\xi < \nabla\xi, \nabla f > dv & (4)\\
&\leq 2\int_{B(x_0,t)\setminus B(x_0,s)} f\xi |\nabla\xi||\nabla f| dv & (5)\\
&\leq 2(\int_{B(x_0,t)\setminus B(x_0,s)} \xi^2 |\nabla f|^2 dv)^{\frac{1}{2}} \cdot (\int_{B(x_0,t)\setminus B(x_0,s)} f^2 |\nabla \xi|^2 dv)^{\frac{1}{2}} & (6)
\end{aligned}
$$

where we apply the Cauchy-Schwarz Inequality in (6).

Therefore, we obtain the following important inequality:

$$
\begin{aligned}
&(\int_{B(x_0,t)} \xi^2 |\nabla f|^2 dv)^2 \\
&\leq \frac{4C_1^2}{(t-s)^2} \int_{B(x_0,t)\setminus B(x_0,s)} \xi^2 |\nabla f|^2 dv \cdot \int_{B(x_0,t)\setminus B(x_0,s)} f^2 dv & (7)
\end{aligned}
$$

Let $\xi = \xi_j(x_0) = \xi(x_0, r_j, r_{j+1})$ and $s = r_j$ and $t = r_{j+1}$ for an increasing sequence $\{r_j\}$ where $r_{j+1} \geq 2r_j$ and $r_j \to \infty$ as $j \to \infty$.

We define

$$
\begin{aligned}
A_j &= \frac{1}{r_j^2} \int_{B(x_0,r_j)} f^2 dv \\
Q_{j+1} &= \int_{B(x_0,r_{j+1})} \xi_j^2 |\nabla f|^2 dv
\end{aligned}
$$

We can rewrite the important inequality (7) as

$$
Q_{j+1}^2 \leq \frac{4C_1^2}{(r_{j+1} - r_j)^2} (Q_{j+1} - Q_j)(r_{j+1}^2 A_{j+1} - r_j^2 A_j) \quad (8)
$$

$$
Q_{j+1}^2 \leq 4C_1^2 (Q_{j+1} - Q_j)(4A_{j+1}) \leq 16C_1^2 (Q_{j+1} - Q_j) A_{j+1} \quad (9)
$$

$$Q_{j+1} \leq 16C_1^2 A_{j+1} \tag{10}$$

because we have $\frac{r_{j+1}^2 A_{j+1} - r_j^2 A_j}{(r_{j+1}-r_j)^2} \leq (r_{j+1}^2 A_{j+1})(\frac{4}{r_{j+1}^2}) = 4A_{j+1}$ due to the fact $r_{j+1} - r_j \geq \frac{1}{2} r_{j+1}$ since $r_{j+1} \geq 2r_j$. So, we have a bounded increasing sequence of $\{Q_{j+1}\}$ because $Q_{j+1} \leq 16C_1^2 A_{j+1} < \infty$ due to the bounded sequence of $A_j < K$ as $j \to \infty$ for $K > 0$ by our assumption $\liminf_{r\to\infty} \frac{\int_{B(r)} f^2 dv}{r^2} < \infty$. We claim that this bounded increasing sequence of $\{Q_{j+1}\}$ must be convergent and $\lim_{j\to\infty} Q_{j+1} = L \geq 0$. By (9), $\forall N > 1$, we find

$$\begin{aligned} & \sum_{j=1}^{N} Q_{j+1}^2 \\ \leq \quad & \sum_{j=1}^{N} 4C_1^2 (Q_{j+1} - Q_j)(4A_{j+1}) \\ \leq \quad & 16C_1^2 K \sum_{j=1}^{N} (Q_{j+1} - Q_j) \\ \leq \quad & 16C_1^2 K (Q_{N+1} - Q_1) \\ \leq \quad & 16C_1^2 K Q_{N+1} \\ \leq \quad & 16C_1^2 K L \end{aligned} \tag{11}$$

Furthermore, we have $\sum_{j=1}^{\infty} Q_{j+1}^2 \leq 16C_1^2 KL < \infty$ where the telescoping series $\sum_{j=1}^{N} (Q_{j+1} - Q_j) = Q_{N+1} - Q_1$. By the limit of n-th term of convergent series, we know $Q_{j+1}^2 \to 0$ as $j \to \infty$ because $\sum_{j=1}^{\infty} Q_{j+1}^2 \leq 16C_1^2 KL < \infty$ is convergent. The conclusion of $Q_{j+1} \to 0$ indicates $\int_M |\nabla f|^2 dv = 0$, that is, $|\nabla f| = 0$ and $\nabla f = 0$. So, f has the constant value (i.e.$f = C$). □

Theorem 4. *Let u be a smooth bi-harmonic map from the domain (M^m, g) to the target space (N^n, h). The domain M is a complete manifold and the target space N is a complete manifold with non-positive sectional curvatures. If its tension field $\tau(u)$ satisfies $\liminf_{r\to\infty} \frac{\int_{B(r)} |\tau(u)|^4 dv}{r^2} < \infty$ (i.e. $|\tau(u)|$ has p-finite growth for $p = 2, q = 4$) then we claim the norm of $\tau(u)$ must be constant, that is, $|\tau(u)| = C$.*

Proof. We know that $\Delta^u = -\overline{\nabla} * \overline{\nabla} = trace_g(\nabla^u \nabla^u)$ is a rough Laplacian on bundles $T^\star M \bigotimes u^{-1}TN$ and $\tau_2(u) = -\overline{\nabla} * \overline{\nabla}\tau(u) + \sum R^N(\tau(u)) = \Delta^u(\tau(u)) + \sum R^N(\tau(u))$. Furthermore, for a bi-harmonic map, since $0 = \tau_2(u) = -\overline{\nabla} * \overline{\nabla}\tau(u) + \sum_{i=1}^m R^N(du(e_i), \tau(u))du(e_i)$, we know that $-\overline{\nabla} * \overline{\nabla}\tau(u) = -\sum_{i=1}^m R^N(du(e_i), \tau(u))du(e_i)$.

$$
\begin{aligned}
& \frac{1}{2}\Delta(|\tau(u)|^2) \\
= \quad & \frac{1}{2}(trace_g \nabla^u \nabla^u) < \tau(u), \tau(u) > && (12) \\
= \quad & \frac{1}{2} trace_g \nabla^u (2 < \nabla^u \tau(u), \tau(u) >) && (13) \\
= \quad & trace_g [< \nabla^u \nabla^u \tau(u), \tau(u) > + < \nabla^u \tau(u), \nabla^u \tau(u) >] && (14) \\
= \quad & < trace_g \nabla^u \nabla^u \tau(u), \tau(u) > + trace_g < \nabla^u \tau(u), \nabla^u \tau(u) > && (15) \\
= \quad & < -\overline{\nabla} * \overline{\nabla}\tau(u), \tau(u) > + \sum_{i=1}^m < \nabla_{e_i}\tau(u), \nabla_{e_i}\tau(u) > && (16) \\
= \quad & < -\sum_{i=1}^m R^N(du(e_i), \tau(u))du(e_i), \tau(u) > + |\nabla^u \tau(u)|^2 && (17) \\
\geq \quad & 0 && (18)
\end{aligned}
$$

where we use the facts:$-\overline{\nabla} * \overline{\nabla}\tau(u) = -\sum_{i=1}^m R^N(du(e_i), \tau(u))du(e_i) \geq 0$ since the target space N has the non-positive sectional curvatures. So, we prove $\Delta(|\tau(u)|^2) \geq 0$. Next, by applying $f = |\tau(u)|^2$ in the Lemma 1, we obtain our conclusion $|\tau(u)|^2 = f = C$ that is, $|\tau(u)|$ is constant. □

3.3. Connections between Bi-Harmonic Maps and Harmonic Maps

Theorem 5. *Let u be a smooth bi-harmonic map from the domain (M^m, g) to the target space (N^n, h). The domain M is a complete Riemannian manifold with $\lim_{r\to\infty} \frac{Vol(B(r))}{r^2} \geq C_2 > 0$. The target space N is a complete manifold with non-positive sectional curvatures. If its tension field $\tau(u)$ satisfies $\liminf_{r\to\infty} \frac{\int_{B(r)} |\tau(u)|^4 dv}{r^2} = 0$ (i.e. $|\tau(u)|$ has the vanishing p-finite growth for $p = 2, q = 4$) then we claim this bi-harmonic map must be a harmonic map, that is, $\tau(u) = 0$.*

Proof. By the proof of Theorem 4, we know that $|\tau(u)|$ must have a constant value. If $|\tau(u)| = C > 0$, we can easily get the result of $\liminf_{r\to\infty} \frac{\int_{B(r)} |\tau(u)|^4 dv}{r^2} =$

$C^4 \lim_{r\to\infty} \frac{Vol(B(r))}{r^2} \geq C^4 C_2 > 0$ which conflicts with an assumption of the vanishing 2-finite growth $\liminf_{r\to\infty} \frac{\int_{B(r)} |\tau(u)|^4 dv}{r^2} = 0$. Therefore, we get $|\tau(u)| = 0$ that is $\tau(u) = 0$. In this case, a bi-harmonic map becomes a harmonic map. □

Remark 1. *Let u be a smooth bi-harmonic map from the domain (M^m, g) to the target space (N^n, h). The domain M is a complete Riemannian manifold with $\lim_{r\to\infty} \frac{Vol(B(r))}{r^2} \geq C_2 > 0$. The target space N is a complete manifold with non-positive sectional curvatures. If its tension field $\tau(u)$ has the finite q-energy in L^q space for $q = 4$ (i.e. $\int_{B(r)} |\tau(u)|^4 dv < \infty$), then we claim this bi-harmonic map must be a harmonic map, that is, $\tau(u) = 0$.*

Proof. We know that the tension field $\tau(u)$ has the finite q-energy in L^q space for $q = 4$ (i.e. $\int_{B(r)} |\tau(u)|^4 dv < \infty$), which indicates $|\tau(u)|$ has the vanishing p-finite growth for $p = 2, q = 4$. By the proof of Theorem 5, we can obtain our result. □

Lemma 2. *For a smooth map $u : (M^m, g) \to (N^n, h)$, we define a vector field on M: $du \cdot \tau(u) = \sum_{i=1}^{m} < du(e_i), \tau(u) >_N e_i$ where $\{e_i\}$ is a local orthonormal frame field on M such that $\nabla_{e_i} e_i = 0$. We claim $|du \cdot \tau(u)| \leq |du||\tau(u)|$ and $div(du \cdot \tau(u)) = |\tau(u)|^2 + < du, \nabla\tau(u) >$.*

Proof.

$$
\begin{aligned}
& div(du \cdot \tau(u)) \\
= & \sum_{i,j=1}^{m} < \nabla_{e_j}(< du(e_i), \tau(u) >_N e_i), e_j > \qquad (19)\\
= & \sum_{i,j=1}^{m} < \big(\nabla_{e_j} < du(e_i), \tau(u) >_N \big) e_i, e_j > \\
& + \sum_{i,j=1}^{m} << du(e_i), \tau(u) >_N \nabla_{e_j} e_i, e_j > \qquad (20)\\
= & \sum_{j=1}^{m} < \big(\nabla_{e_j} < du(e_j), \tau(u) >_N \big) e_j, e_j > \qquad (21)\\
= & \sum_{j=1}^{m} < \nabla_{e_j} du(e_j), \tau(u) >_N + \sum_{j=1}^{m} < du(e_j), \nabla_{e_j} \tau(u) >_N \qquad (22)\\
= & < \tau(u), \tau(u) > + \sum_{j=1}^{m} < du(e_j), \nabla_{e_j} \tau(u) > \qquad (23)\\
= & |\tau(u)|^2 + < du, \nabla \tau(u) > \qquad (24)
\end{aligned}
$$

where we use the facts $\nabla_{e_j} e_j = 0, < e_i, e_j >= 0, i \neq j, < e_j, e_j >= 1$, and $< \nabla_{e_j} e_i, e_j >= \nabla_{e_j} < e_i, e_j > - < e_i, \nabla_{e_j} e_j >= \nabla_{e_j} < e_i, e_j >= 0$ since

$$
\nabla_{e_j} < e_i, e_j >= \begin{cases} \nabla_{e_j} 0 = 0 & \text{when } i \neq j, < e_i, e_j >= 0, \\ \nabla_{e_j} 1 = 0 & \text{when } i = j, < e_i, e_i >= 1. \end{cases}
$$

□

Theorem 6. *Suppose a smooth map u from the domain (M^m, g) to the target space (N^n, h) has the parallel tension field at a point $x_0 \in M$ (i.e.$\nabla \tau(u) = 0$) and satisfies $\liminf_{r \to \infty} \frac{\int_{B(x_0,r)} |du|^2 dv}{r^2} < \infty$ (i.e. $|du|$ has the p-finite growth for $p = 2, q = 2$), then we claim this map must be a harmonic map, that is, $\tau(u) = 0$.*

Proof. We define a rotationally symmetric Lipschitz continuous function $\xi(x) = \xi(x_0, s, t)$ and a constant $C_1 > 0$ (independent of x_0, s, t) for $0 < s < t$ on M with the following properties:

(i) $\xi = 1$ on $B(x_0, s)$

(ii) $\xi = 0$ off $B(x_0, t)$

(iii) $0 < \xi < 1$ on $B(x_0, t) \backslash B(x_0, s)$

(iv) $|\nabla \xi| \leq \frac{C_1}{t-s}$ a.e. on M.

$$
\begin{aligned}
& \int_{B(x_0,t)} \xi^2 |\tau(u)|^2 dv \\
= & \int_{B(x_0,t)} \xi^2 |\tau(u)|^2 dv + \int_{B(x_0,t)} \xi^2 < du, \nabla \tau(u) > dv \qquad (25) \\
= & \int_{B(x_0,t)} \xi^2 div(du \cdot \tau(u)) dv \qquad (26) \\
= & -\int_{B(x_0,t)\backslash B(x_0,s)} 2\xi < du \cdot \tau(u), \nabla \xi > dv \qquad (27) \\
\leq & \frac{2C_1}{t-s} \Big(\int_{B(x_0,t)\backslash B(x_0,s)} |du|^2 dv \Big)^{\frac{1}{2}} \cdot \Big(\int_{B(x_0,t)\backslash B(x_0,s)} \xi^2 |\tau(u)|^2 dv \Big)^{\frac{1}{2}} \qquad (28)
\end{aligned}
$$

where we use the facts $div(du \cdot \tau(u)) = |\tau(u)|^2 + < du, \nabla \tau(u) >$ (see Lemma 2) in (26), Integration by Parts in (27), $|du \cdot \tau(u)| \leq |du||\tau(u)|$ and Cauchy-Schwarz Inequality in (28).

We can rewrite the above inequality (28) as

$$\tilde{Q}_{j+1} \leq \frac{2C_1}{r_{j+1} - r_j} (\tilde{Q}_{j+1} - \tilde{Q}_j)^{\frac{1}{2}} (r_{j+1}^2 \tilde{A}_{j+1} - r_j^2 \tilde{A}_j)^{\frac{1}{2}} \qquad (29)$$

where we let $\xi = \xi_j(x_0) = \xi(x_0, r_j, r_{j+1})$ and $s = r_j$ and $t = r_{j+1}$ for an increasing sequence $\{r_j\}$ such that $r_{j+1} \geq 2r_j$, $r_j \to \infty$ as $j \to \infty$ and define:

$$
\begin{aligned}
\tilde{A}_j &= \frac{1}{r_j^2} \int_{B(x_0, r_j)} |du|^2 dv \\
\tilde{Q}_{j+1} &= \int_{B(x_0, r_{j+1})} \xi_j^2 |\tau(u)|^2 dv
\end{aligned}
$$

Furthermore, we obtain the following inequalities:

$$\tilde{Q}_{j+1}^2 \leq \frac{4C_1^2}{(r_{j+1} - r_j)^2} (\tilde{Q}_{j+1} - \tilde{Q}_j)(r_{j+1}^2 \tilde{A}_{j+1} - r_j^2 \tilde{A}_j) \qquad (30)$$

$$\tilde{Q}_{j+1}^2 \leq 4C_1^2 (\tilde{Q}_{j+1} - \tilde{Q}_j)(4\tilde{A}_{j+1}) \leq 16 C_1^2 (\tilde{Q}_{j+1} - \tilde{Q}_j) \tilde{A}_{j+1} \qquad (31)$$

$$\tilde{Q}_{j+1} \leq 16 C_1^2 \tilde{A}_{j+1} \qquad (32)$$

because we have $\frac{r_{j+1}^2 \tilde{A}_{j+1} - r_j^2 \tilde{A}_j}{(r_{j+1}-r_j)^2} \leq (r_{j+1}^2 \tilde{A}_{j+1})(\frac{4}{r_{j+1}^2}) = 4\tilde{A}_{j+1}$ due to the fact $r_{j+1} - r_j \geq \frac{1}{2} r_{j+1}$ since $r_{j+1} \geq 2r_j$. So, we have a bounded increasing sequence of $\{\tilde{Q}_{j+1}\}$ because $\tilde{Q}_{j+1} \leq 16C_1^2 \tilde{A}_{j+1} < \infty$ due to the bounded sequence of $\tilde{A}_j < \tilde{K}$ as $j \to \infty$ for $\tilde{K} > 0$ by our assumption $\liminf_{r\to\infty} \frac{\int_{B(r)} |du|^2 dv}{r^2} < \infty$. We claim that this bounded increasing sequence of $\{\tilde{Q}_{j+1}\}$ must be convergent and $\lim_{j\to\infty} \tilde{Q}_{j+1} = \tilde{L} \geq 0$.

By the similar proof in Lemma 1, we can draw our conclusion: $\lim_{j\to\infty} \tilde{Q}_{j+1} = 0$ due to the convergent series $\sum_{j=0}^{\infty} \tilde{Q}_{j+1}^2 \leq 16C_1^2 \tilde{K}\tilde{L} < \infty$ when $\liminf_{j\to\infty} \tilde{A}_j = \liminf_{j\to\infty} \frac{1}{r_j^2} \int_{B(x_0,r_j)} |du|^2 dv \leq \tilde{K} < \infty$. More precisely, we claim that $\int_M |\tau(u)|^2 dv = 0$ and $|\tau(u)| = 0$ which indicates $\tau(u) = 0$ a.e. on M. □

4. Conclusions

Identifying robust algorithms for energy estimating techniques to detect infinite energy in various manifold structures has emerged as a complex and significant research topic among scholars. The original work in our research findings is to break the constraints of the fundamental limitation of the finite q-energy technique. Our innovative p-balanced energy algorithms can detect q-energy growth approaching infinity by extending from the finite q-energy in L^q spaces to infinite q-energy in non-L^q spaces. In order to address the inherent challenges associated with limited q-energy detection, our proposed p-balanced energy strategy offers a promising approach towards achieving successful detection of infinite q-energy.

The p-balanced energy technique provides us with a powerful tool to find a rigid dichotomy between infinity energy and finite constant energy when we explore connections between bi-harmonic maps and harmonic maps on curved manifolds. More precisely, we can classify the energy into two distinct categories: either energy at an infinite level or energy with the finite constant value. Furthermore, we can investigate what kinds of curved manifold structures would make bi-harmonic maps become harmonic maps.

Regarding algorithms in the p-balanced energy growth technique, we provide the theoretic justification of the technical breakthroughs. Calculation skills such as Cauchy-Schwarz Inequality, Divergence Theorem, Integration by Parts, and Convergence Tests for Infinite Series have been used to evaluate limit and integration for the norm of tension fields in curved manifolds. This innovative p-energy algorithm has been successfully applied to the norm of differential forms on Manifolds with the support of Poincare-Sobolev Inequality in order to solve Liouville-type problems [16, 17, 19, 20, 21, 18].

5. Future Research Plan

The p-balanced energy technique encompasses five distinct cases: "p-finite, p-mild, p-obtuse, p-moderate, and p-small". Our study primarily examines the initial instance of p-finite. Our ongoing research will focus on investigations of the other four cases to complete the p-balanced energy approach.

Applications of p-balanced energy technique to $\mathcal{A}$-harmonic theory will become our future research subject. $\mathcal{A}$-Harmonic equations are partial differential equations involved with any harmonic operators in general settings, including p-harmonic operators and harmonic operators as special examples [10]. $\mathcal{A}$-Harmonic theory can be found in engineering mechanics, elasticity, fluid pressure, fluid flow motion, etc. As we all know, bi-harmonic equations are widely used in modeling structures that react elastically to external forces [5, 7] and are also utilized for motion equations such as Stokes flow (i.e., creeping motion) [2, 4, 9, 8].

The accuracy of our theoretical justification and the efficiency of our new energy technique in this project will provide an innovative approach to studying $\mathcal{A}$-harmonic theory on curved manifolds with a mix of curvature signs.

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