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ON FIBONACCI-CORDIAL LABELING OF TWO SPECIAL KINDS OF GRAPHS

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On Fibonacci-Cordial Labeling of Two Special Kinds of Graphs

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Abstract

In this paper we investigate Fibonacci-Cordial labeling of graphs in two special families of "corona products", namely, $C_n \bigodot N_m$ and $C_n \bigodot K_m$. We prove the existence of the Fibonacci-Cordial labeling of $C_n \bigodot N_m$ and establish an efficient algorithm to find one. We also prove the nonexistence of the Fibonacci-Cordial labeling for $C_n \bigodot K_m$ when m is much larger than n.

1 Introduction

Graph labeling is an interesting research topic. Readers can find an extensive and updated survey on the research of graph labeling in [3]. Cahit [2] introduced the concept of cordial labeling. Rokad and Ghodasara [1] introduced the concept of Fibonacci Cordial Labeling.

As discussed in [6], the following definition of Fibonacci-Cordial labeling is actually the same as defined by Rokad and Ghodasara in [1]

Definition 1.1. *An injective function $f : V(G) \longrightarrow \{F_0, F_1, F_2, ..., F_n\}$ where F_j is the jth Fibonacci number (j = 0, 1, 2, ..., n, $n = |V(G)|$) is said to be Fibonacci cordial labeling if the induced function $f^* : E(G) \longrightarrow \{0, 1\}$ defined by $f^*(uv) = (f(u) + f(v))(mod 2)$ satisfies the condition $|e_f(0) - e_f(1)| \leq 1$. Here $e_f(i)$, i = 0 or 1 as the number of edges with induced label i. A graph which admits Fibonacci cordial labeling is called Fibonacci cordial graph.*

We define Corona Products of two graphs as follows [4].

Definition 1.2. *The corona product of two graphs H and J, denoted $H \bigodot J$, is the graph formed from one copy of H and $|V(H)|$ copies of J where the i^{th} vertex of H is adjacent to every vertex in the i^{th} copy of J*

In this paper, we consider two families of corona products. In both families, H is a cycle graph with n vertices, namely, C_n, where $n \geq 3$. In the first family, J is a null graph with m vertices, namely, N_m. In the second family, J is a complete graph with m vertices, namely, K_m. Sometimes, the researchers refer N_m as the complement of K_m, and denote it as $\overline{K_m}$, as in [5].

We prove that the graph $G = C_n \bigodot N_m$ is always Fibonacci Cordial for any $n \geq 3$ and $m \geq 1$. We also implement an efficient algorithm to find a Fibonacci Cordial labeling for $G = C_n \bigodot N_m$.

We prove that the graph $G = C_n \bigodot K_m$ is not Fibonacci Cordial when m is much bigger than n.

2 Fibonacci Cordial Labeling for $G = C_n \bigodot N_m$

Definition 2.1. *For a given positive integer n, define the binary sequence* $B_n = \{b_0, b_1, b_2, b_3, ..., b_n\}$ *as follows:*

$$b_i = \begin{cases} 0 \ \textit{if } i \equiv 0 \ \textit{mod } 3 \\ 1 \ \textit{otherwise} \end{cases} \tag{1}$$

It is clear that the i^{th} element in the binary sequence B_n is the parity of the i^{th} element in the Fibonacci sequence $\{F_0, F_1, ..., F_n\}$. That is $b_i \equiv F_i \ mod \ 2$ for $i = 0, 1, ..., n$. Since the criteria of Fibonacci Cordial labeling is about the parity of the induced labels of the edges of the graph, we can simply replace each Fibonacci number in labeling by its parity to achieve the same induced labels of the edges. Therefore, when we study Fibonacci Cordial Labeling, we can replace the label sequence $\{F_0, F_1, F_2, ..., F_n\}$ in definition 1.1 by the binary sequence B_n defined in 2.1.

Lemma 2.1. *In binary sequence B_n, the number of 0s is* $\lfloor \frac{n}{3} \rfloor + 1$

Proof. B_n is corresponding to parities of Fibonacci sequence $\{F_0, F_1, ..., F_n\}$. By the construction of the Fibonacci sequence, the lemma is self proven. □

Theorem 2.2. *For any $n \geq 3$ and $m \geq 1$, the graph $G = C_n \bigodot N_m$ is Fibonacci Cordial*

Proof. Let's denote the vertices in C_n as $c_1, c_2, ..., c_n$ such that the edges are $c_1c_2, c_2c_3, c_3c_4, ..., c_nc_1$. Denote the i^{th} copy of N_m, which is connected with c_i, as N_m^i. We denote the vertices of N_m^i as $n_{i1}, ..., n_{im}$. It is clear that G has $n(m+1)$ vertices and $n(m+1)$ edges. The binary label sequence is $B_{n(m+1)}$, which has $1 + n(m+1)$ elements, and there are $z = \lfloor \frac{n(m+1)}{3} \rfloor + 1$ many zeros and $n(m+1) + 1 - z$ many ones

We reserve one 0, say b_0, and use the remaining $n(m+1)$ binary digits to label G. First, we label all vertices in C_n with 1s. Then we start with N_m^1, label every other copy of N_m^i with all zeros until no 0s are left. At this point, all vertices in $N_m^1, N_m^3, N_m^5, ..., N_m^{2k-1}$ are labeled with zeros where $k = \lfloor \frac{z-1}{m} \rfloor = \lfloor \frac{\lfloor \frac{n(m+1)}{3} \rfloor}{m} \rfloor$. It is possible that there are some vertices in N_m^{2k+1} that are labeled with 0. Then we label all the rest of vertices with ones.

With this initial labeling, it is easy to see that we have more edges with induced label 0. In fact, there are $n(m+1)$ edges, and about $\frac{1}{3}$ of them have induced label 1. The number of induced zeros is about $\frac{n(m+1)}{3}$ more. We can start the following procedure: change label c_2 to be 0 and change one label of N_m^1, say n_{11} be 1. If we do so, we get m+1 more ones. So the difference between zeros and ones is reduced by 2(m+1). If it is needed, we do the same thing to N_m^3 and a_4, then N_m^5 and a_6, and so on until the number of induced zeros is less than the number of induced ones. Now we know the difference will be at most 2(m+1).

Then we begin to exchange label 0 of N_m^1 and label 1 of N_m^2. In each such exchange, we will increase the number of induced 0s by 4. If there are no more label 0s in N_1^1, we repeat this between N_m^3 and N_m^4, and so on. We keep doing this until the number of induced zeros is less than the number of induced ones by 4, 3, 2, or 1. If by 4, do it one more time, we are all set since we will get equal number of induced zeros and ones. If by 3, do it one more time, we are also set since the number of induced zeros is one more that the number of induced ones. If by 1, we don't need to do anything. If by 2, then we can replace a label of 1 in a copy of N_m that is attached to a vertex labeled 0 with 0. Then the number of induced zeros and the number of induced ones will be equal. □

In fact, the above proof gives us an algorithm to find a Fibonacci Cordial labeling for $C_n \bigodot N_m$. We write a small program using python, which takes n and m as inputs, and produces a Fibonacci Cordial labeling of $C_n \bigodot N_m$. Here are a couple of sample runs.

Sample Run One: $C_4 \bigodot N_2$. The output is:

$$[1, 0, 1, 1]\ [[1, 0], [0, 1], [0, 0], [1, 1]]\ 6,\ 6$$

The output is interpreted as: The number of induced zeros is 6; the number of induced ones is 6. a_1, a_2, a_3, a_4 are labeled with 1, 0, 1, 1. The N_2^1 is labeled with 1, 0; N_2^2 is labeled with 0, 1; N_2^3 is labeled with 0, 0; and N_2^4 is labeled with 1, 1. It is easy to check manually that this is a Fibonacci Cordial labeling.

Sample Run Two: $C_7 \bigodot N_4$. The output is:

$$[1, 0, 1, 0, 1, 1, 1]$$

$$[[1, 0, 0, 0], [1, 1, 1, 1], [1, 1, 1, 0], [1, 0, 0, 1], [1, 1, 1, 1], [1, 1, 1, 1], [0, 0, 0, 1]]\ 18,\ 17$$

The output is interpreted as: The number of induced zeros is 18, and the number of induced ones is 17. $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ is labeled with 1, 0, 1, 0, 1, 1, 1. N_4^1 is labeled with 1, 0, 0, 0. N_4^3 is labeled with 1, 1, 1, 0; N_4^4 is labeled with 1, 0, 0, 1; and N_4^7 is labeled with 0, 0, 0, 1. The rest of the copies of N_4 are labeled with 1, 1, 1, 1.

3 Fibonacci Cordial Labeling for $G = C_n \bigodot K_m$

In [4], the authors studied the Fibonacci Cordial Labeling of $C_n \bigodot K_m$. In this paper, we prove the following theorem.

Theorem 3.1. *For any given $n \geq 3$, when m is big enough, $C_n \bigodot K_m$ is not Fibonacci Cordial.*

Proof. We simply show that the maximum possible induced ones are too less when compared with the induced zeros. Therefore it is not possible to make their difference is less than or equal to 1. Hence there is no Fibonacci Cordial labeling.

Suppose we label x_k many vertices in K_m that attached to vertex a_k in C_n. Then, the number of induced ones in G will be at most $\sum_{k=1}^{n} x_k(m - x_k) + mn$ with the constraint that $x_1 + x_2 + ... + x_n \leq \frac{n(m+1)}{3}$. Let's estimate the maximum of $\sum_{k=1}^{n} x_k(m - x_k)$ with constrain that $x_1 + x_2 + ... + x_n = \frac{n(m+1)}{3}$. It is easy to see when all x_i are equals it achieves the maximum. So the maximum is $\frac{2m^2n}{9}$. Hence, there are at most $\frac{2m^2n}{9} + mn = \frac{2m^2n}{9} + O(m)$ induced ones. However, the total induces edge labels are $n + mn + \frac{m(m-1)n}{2} = \frac{m^2n}{2} + O(m)$. So the number of induced zeros is $\frac{5m^2n}{18} + O(m)$. So the difference between the numbers of induced 0s and 1s is $\frac{m^2n}{18} + O(m)$, Which goes to infinity as m goes to infinity. □

4 Future Work and Acknowledgements

Theorem 3.1 definitely can be explored more. For instance, we can study for what n and m, $C_n \bigodot K_m$ is Fibonacci Cordial. [4] and [5] already have covered some special cases of n and m. Based on Theorem 3.1, it is natural to ask, given n, what is the largest m, such that $C_n \bigodot K_m$ is Fibonacci Cordial.
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References

[1] A.H.Rokad and G.V.Ghodasara. Fibonacci cordial labeling of some special graphs. *Annals of Pure and Applied Mathematics*, 11(1):133–144, 2016.

[2] Cahit. Cordial graphs, a weaker version of graceful and harmonic graphs. *Ars Combinatoria*, 23:201–207, 1987.

[3] Joseph A. Gallian. A dynamic survey of graph labeling. *Electronic Journal of Combinatorics*, 1, 2018.

[4] Sarbari Mitra and Soumya Bhoumik. Fibonacci cordial labeling of some special families of graphs. *Annals of Pure and Applied Mathematics*, 21(2):135–140, 2020.

[5] Sarbari Mitra and Soumya Bhoumik. Fibonacci cordial labeling of some special families of graphs. *Submitted*, 2024.

[6] Hong Biao Zeng. On k-cordial labeling. *2022 STEM/STEAM and Education PROCEEDINGS*, 2022.